CPSC 221
Basic Algorithms and Data Structures

Asymptotic Analysis

Textbook References:
Koffman: 2.6
EPP 3rd edition: 9.2 and 9.3
EPP 4th edition: 11.2 and 11.3

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CPSC 221: A journey filled with data structures, complexity, and algorithms

- We've talked about ADT
  - Stacks & Queues
- We've talked about arrays and Linked Lists
  - And how to implement Stacks & Queues with them
- But if we have different implementations of an algorithm/ADT, how can we compare them?

Learning Goals

- Define which program operations we measure in an algorithm in order to approximate its efficiency.
- Define "input size" and determine the effect (in terms of performance) that input size has on an algorithm.
- Give examples of common practical limits of problem size for each complexity class.
- Give examples of tractable, intractable problems.
- Given code, write a formula which measures the number of steps executed as a function of the size of the input (N).

Continued…

Learning Goals

- Compute the worst-case asymptotic complexity of an algorithm (i.e., the worst possible running time based on the size of the input (N)).
- Categorize an algorithm into one of the common complexity classes.
- Explain the differences between best-, worst-, and average-case analysis.
- Describe why best-case analysis is rarely relevant and how worst-case analysis may never be encountered in practice.
- Given two or more algorithms, rank them in terms of their time and space complexity.

Continued…

Learning Goals

- Define big-O, big-Omega, and big-Theta: $O(\bullet)$, $\Omega(\bullet)$, $\Theta(\bullet)$
- Explain intuition behind their definitions.
- Prove one function is big-O/Omega/Theta of another function.
- Simplify algebraic expressions using the rules of asymptotic analysis.
- List common asymptotic complexity orders, and how they compare.

A Task to Solve and Analyze

- Find a student’s name in a class given her student ID

Efficiency

- Complexity theory addresses the issue of how *efficient* an algorithm is, and in particular, how well an algorithm *scales* as the problem size increases.
- Some measure of efficiency is needed to compare one algorithm to another (assuming that both algorithms are correct and produce the same answers). Suggest some ways of how to measure efficiency.
  - Time (How long does it take to run?)
  - Space (How much memory does it take?)
  - Other attributes?
    - Expensive operations, e.g. I/O
    - Elegance, Cleverness
    - Energy, Power
    - Ease of programming, legal issues, etc.

Analyzing Runtime

```
old2 = 1;
old1 = 1;
for (i=3; i<n; i++) {
    result = old2+old1;
    old1 = old2;
    old2 = result;
}
```

How long does this take?

Analyzing Runtime

```
old2 = 1;
old1 = 1;
for(i=3; i<n; i++){
  result = old2+old1;
  old1 = old2;
  old2 = result;
}
```

Wouldn't be nice if didn't depend on so many things?

How long does this take?

IT DEPENDS

- What is n?
- What machine?
- What language?
- What compiler?
- How was it programmed?

Number of Operations

- Let us focus on one complexity measure: the **number of operations** performed by the algorithm on an input of a given **size**.
- What is meant by "number of operations"?
  - # instructions executed
  - # comparisons
- Is the "number of operations" a precise indicator of an algorithm's running time (time complexity)? Compare a "shift register" instruction to a "move character" instruction, in assembly language.
  - No, some operations are more costly than others
- Is it a fair indicator?
  - Good enough

Analyzing Runtime

```
old2 = 1
old1 = 1
for(i=3; i<n; i++){
  result = old2+old1
  old1 = old2
  old2 = result
}
```

How many operations does this take?

IT DEPENDS

- What is n?

- Running time is a function of n such as T(n)
- This is really nice because the runtime analysis doesn't depend on hardware or subjective conditions anymore

Input Size

- What is meant by the input size *n*? Provide some application-specific examples.
  - Dictionary:
    - # words
  - Restaurant:
    - # customers or # food choices or # employees
  - Airline:
    - # flights or # luggage or # costumers

- We want to express the number of operations performed as a function of the input size *n*.

Run Time as a Function of **Size of** Input

- But, **which** input?
  - Different inputs of same size have different run times

  E.g., what is run time of linear search in a list?
  - If the item is the first in the list?
  - If it's the last one?
  - If it's not in the list at all?

What should we report?

Which Run Time?

There are different kinds of analysis, e.g.,

- Best Case
- Worst Case
- Average Case (Expected Time)
- Common Case
- etc.

Which Run Time?

There are different kinds of analysis, e.g.,

- Best Case
- Worst Case
- Average Case (Expected Time)
- Common Case
- etc.

Mostly useless

Which Run Time?

There are different kinds of analysis, e.g.,

- Best Case
- Worst Case
- Average Case (Expected Time)
- Common Case
- etc.

Useful, pessimistic

Which Run Time?

Useful, hard to do right

- Average Case (Expected Time)
  - Some problems may be intractable in the worst case, but tractable on average
  - Allows discriminating among algorithms with the same worst case complexity
    - Classic example: MergeSort vs QuickSort

Requires a notion of an "average" input to an algorithm

Uses a probability distribution over input

Which Run Time?

There are different kinds of analysis, e.g.,

- Best Case
- Worst Case
- Average Case (Expected Time)
- Common Case
- etc.

Very useful, but ill-defined

Scalability!

- What's more important?
  - At n=5, plain recursion version is faster.
  - At n=35, complex version is faster.
- Computer science is about solving problems people couldn't solve before. Therefore, the emphasis is almost always on solving the big versions of problems.
- (In computer systems, they always talk about "scalability", which is the ability of a solution to work when things get really big.)

Asymptotic Analysis

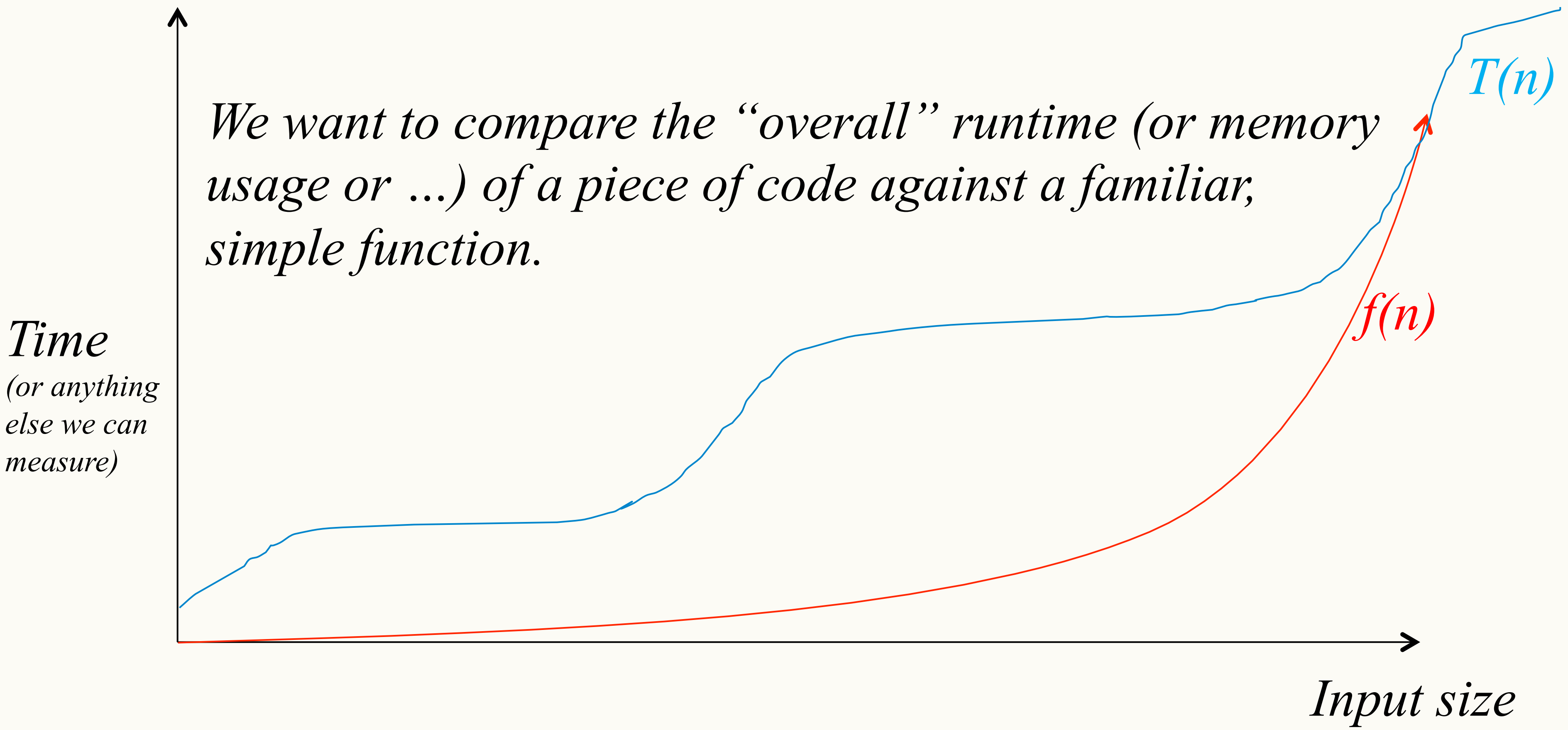
- Asymptotic analysis is analyzing what happens to the run time (or other performance metric) as the input size n goes to infinity.
  - The word comes from "asymptote", which is where you look at the limiting behavior of a function as something goes to infinity.
- This gives a solid mathematical way to capture the intuition of emphasizing scalable performance.
- It also makes the analysis a lot simpler!

Big-O (Big-Oh) Notation

- Let $T(n)$ and $f(n)$ be functions mapping $Z^+ \to R^+$.

Positive real numbers

Positive integers



We want to compare the "overall" runtime (or memory usage or …) of a piece of code against a familiar, simple function.

Big-O Notation

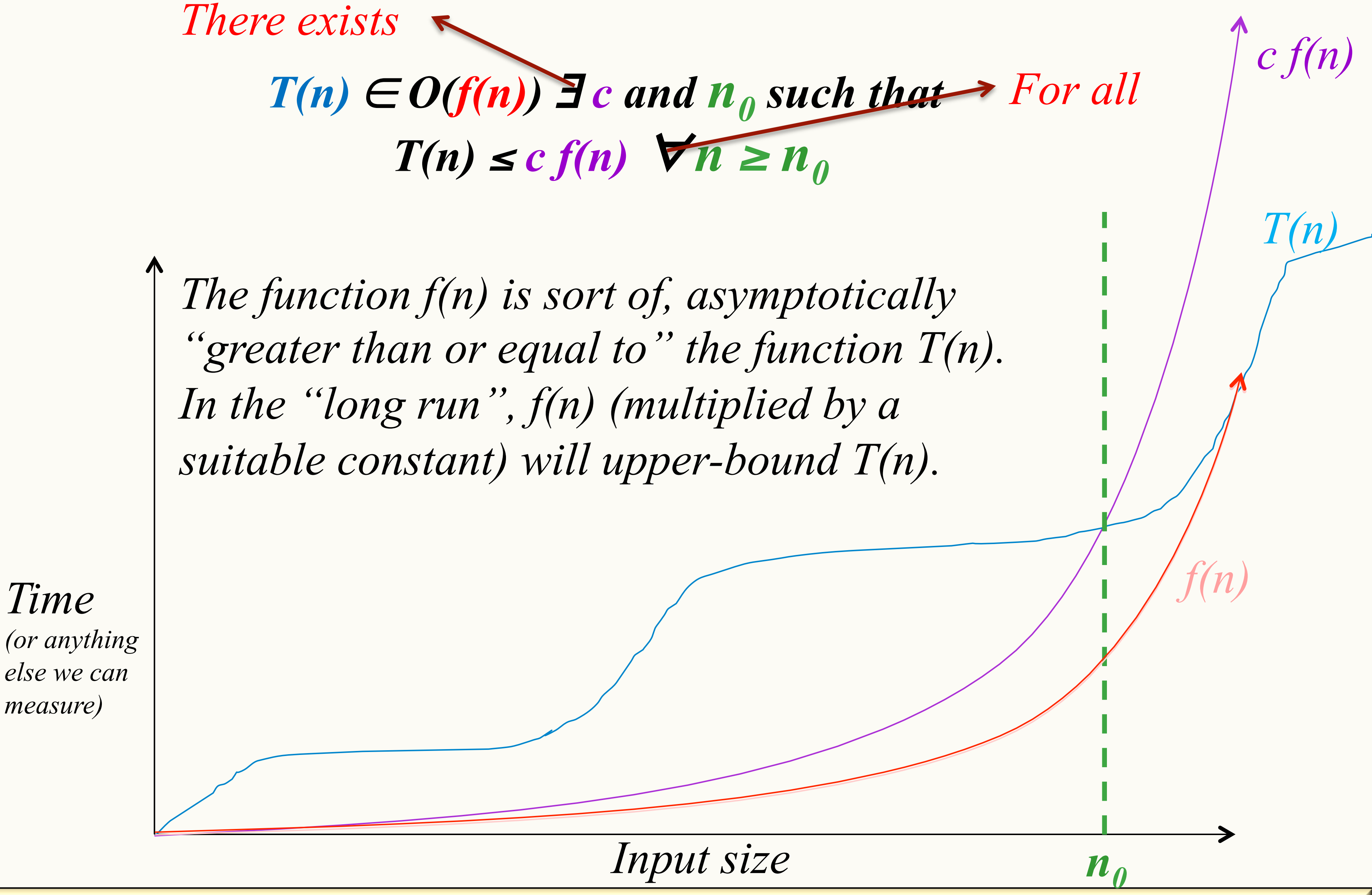
There exists

$T(n) \in O(f(n))$ $\exists$ c and n_0 such that

$T(n) \le c\,f(n)$ $\forall n \ge n_0$

For all

The function f(n) is sort of, asymptotically "greater than or equal to" the function T(n). In the "long run", f(n) (multiplied by a suitable constant) will upper-bound T(n).



Big-O Notation

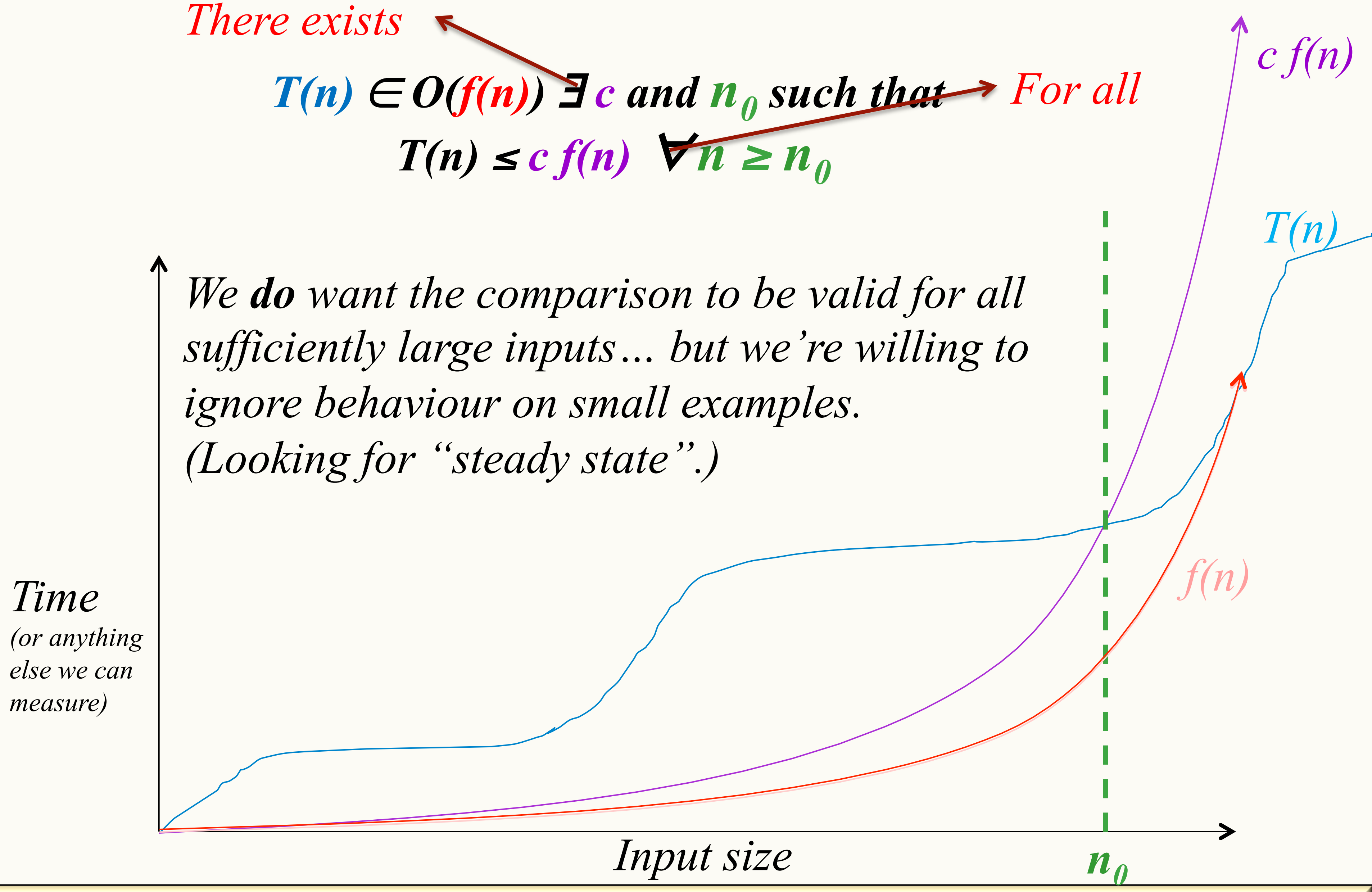
There exists

$T(n) \in O(f(n))$ $\exists$ c and n_0 such that

For all

$T(n) \leq c\, f(n)\ \forall n \geq n_0$

We **do** want the comparison to be valid for all sufficiently large inputs… but we're willing to ignore behaviour on small examples. (Looking for "steady state".)



Big-O Notation (cont.)

- Using Big-O notation, we might say that Algorithm A "runs in time Big-O of $n \log n$", or that Algorithm B "is an order n-squared algorithm". We mean that the number of operations, as a function of the input size n, is $O(n \log n)$ or $O(n^2)$ for these cases, respectively.
- Constants don't matter in Big-O notation because we're interested in the asymptotic behavior as n grows arbitrarily large; but, be aware that large constants can be very significant in an actual implementation of the algorithm.

Rates of Growth

- Suppose a computer executes 10^{12} ops per second:

n =	**10**	**100**	**1,000**	**10,000**	**10^{12}**
n	10^{-11}s	10^{-10}s	10^{-9}s	10^{-8}s	1s
n lg n	10^{-11}s	10^{-9}s	10^{-8}s	10^{-7}s	40s
n^2	10^{-10}s	10^{-8}s	10^{-6}s	10^{-4}s	10^{12}s
n^3	10^{-9}s	10^{-6}s	10^{-3}s	1s	10^{24}s
2^n	10^{-9}s	10^{18}s	10^{289}s		

10^4s = 2.8 hrs ***10^{18}s = 30 billion years***

Asymptotic Analysis Hacks

- Eliminate low order terms
  - $4n + 5 \Rightarrow 4n$
  - $0.5\ n \log n - 2n + 7 \Rightarrow 0.5\ n \log n$
  - $2^n + n^3 + 3n \Rightarrow 2^n$
- Eliminate coefficients
  - $4n \Rightarrow n$
  - $0.5\ n \log n \Rightarrow n \log n$
  - $n \log (n^2) = 2\ n \log n \Rightarrow n \log n$

Silicon Downs

Post #1	*Post #2*
$n^3 + 2n^2$	$100n^2 + 1000$
$n^{0.1}$	$\log n$
$n + 100n^{0.1}$	$2n + 10 \log n$
$5n^5$	$n!$
$n^{-15}2^n/100$	$1000n^{15}$
$8^{2\lg n}$	$3n^7 + 7n$
mn^3	$2^m n$

For each race, which "horse" is "faster". Note that faster means smaller, not larger!

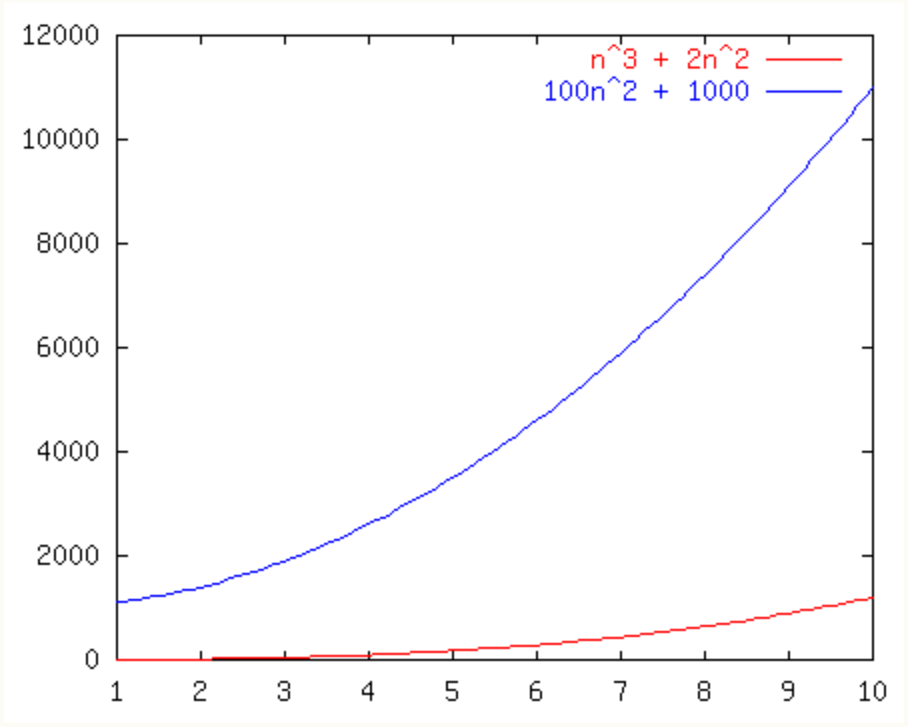
All analysis are done asymptotically

a. *Left*
b. *Right*
c. *Tied*
d. *It depends*

Race I

a. Left
b. Right
c. Tied
d. It depends

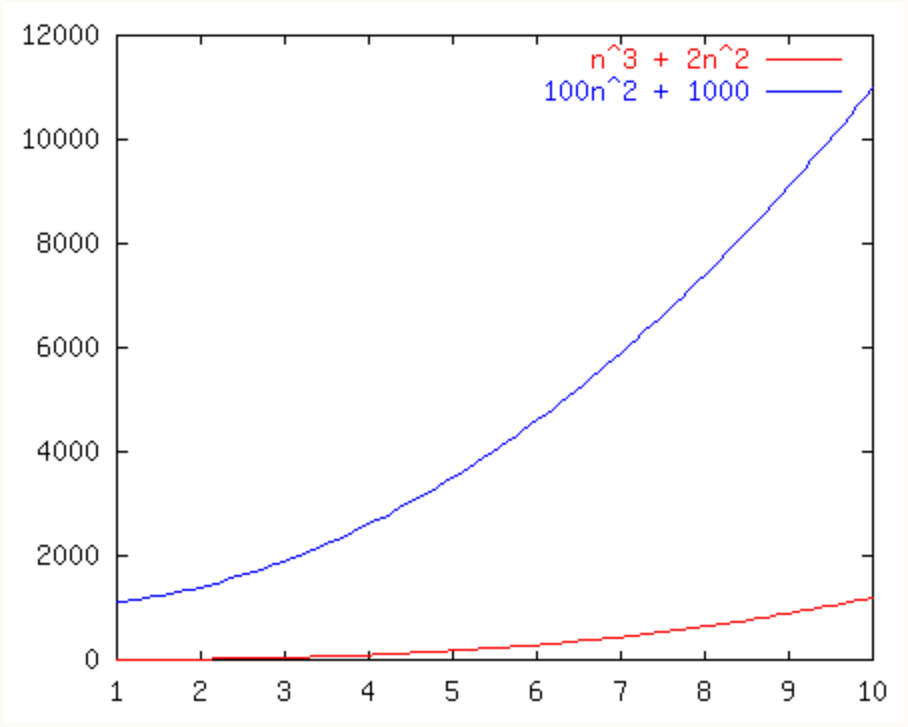
$n^3 + 2n^2$ vs. $100n^2 + 1000$

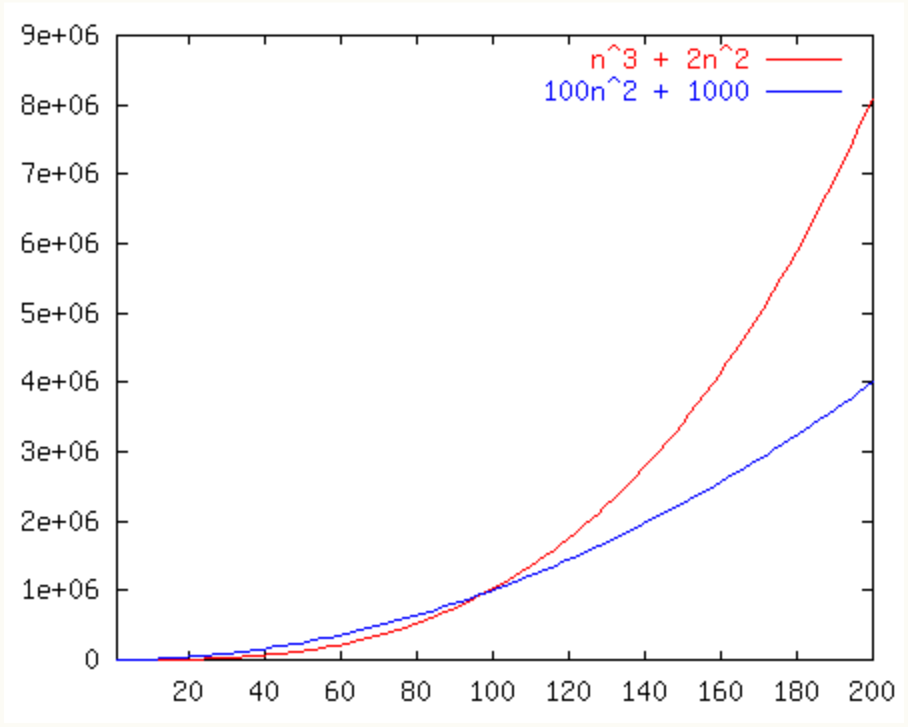


Race I

a. Left
b. Right
c. Tied
d. It depends

$n^3 + 2n^2$ vs. $100n^2 + 1000$

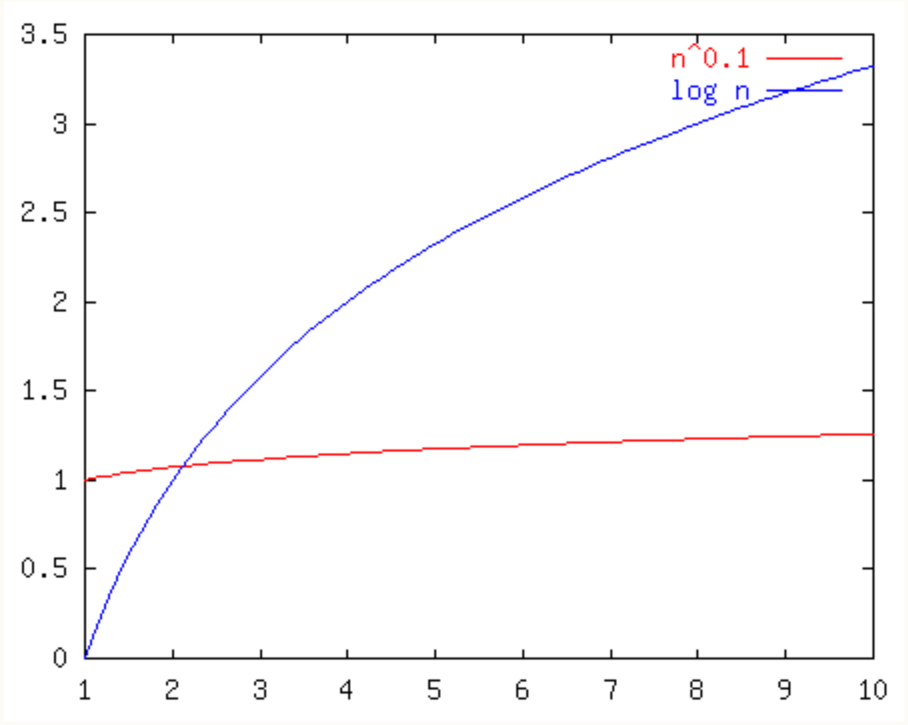




Race II

a. Left
b. Right
c. Tied
d. It depends

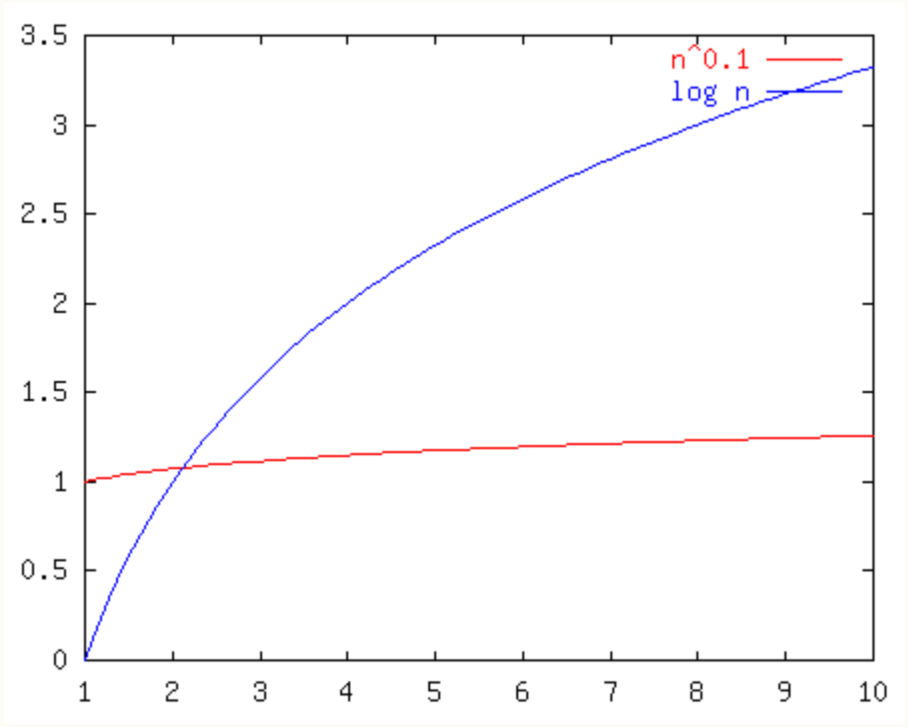
$n^{0.1}$ vs. $\log n$

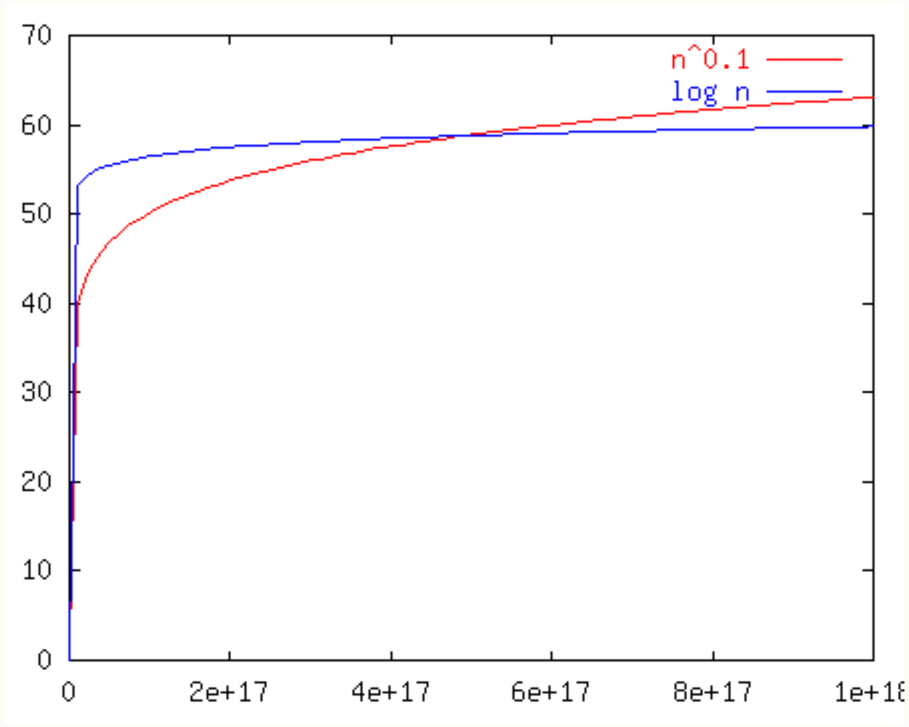


Race II

a. *Left*
b. *Right*
c. *Tied*
d. *It depends*

$n^{0.1}$ *vs.* **log n**



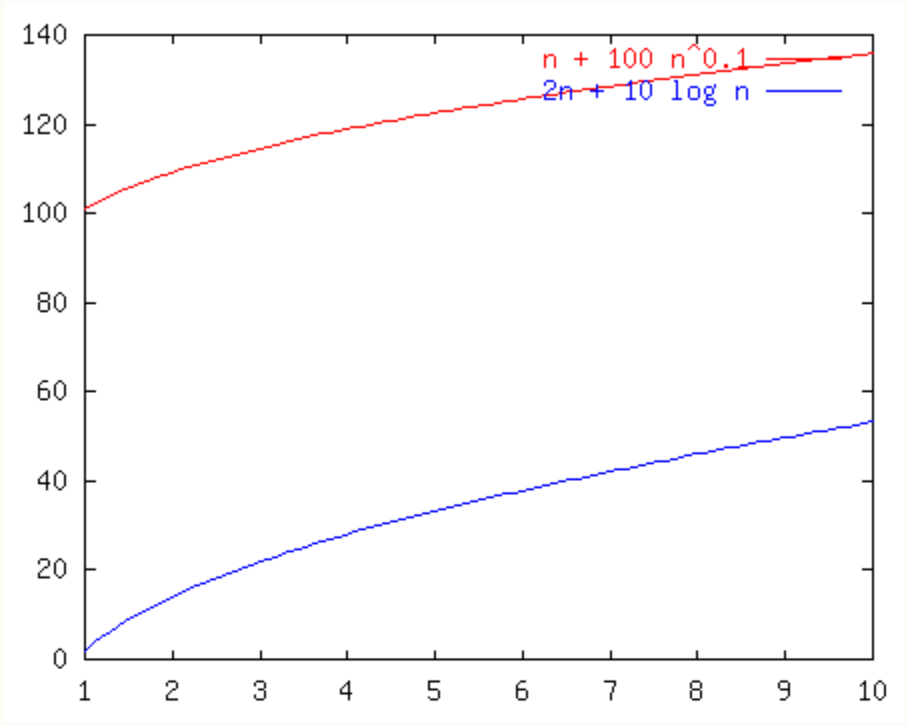


Moral of the story? n^{eps} is slower than log n for any eps > 0

Race III

a. Left
b. Right
c. Tied
d. It depends

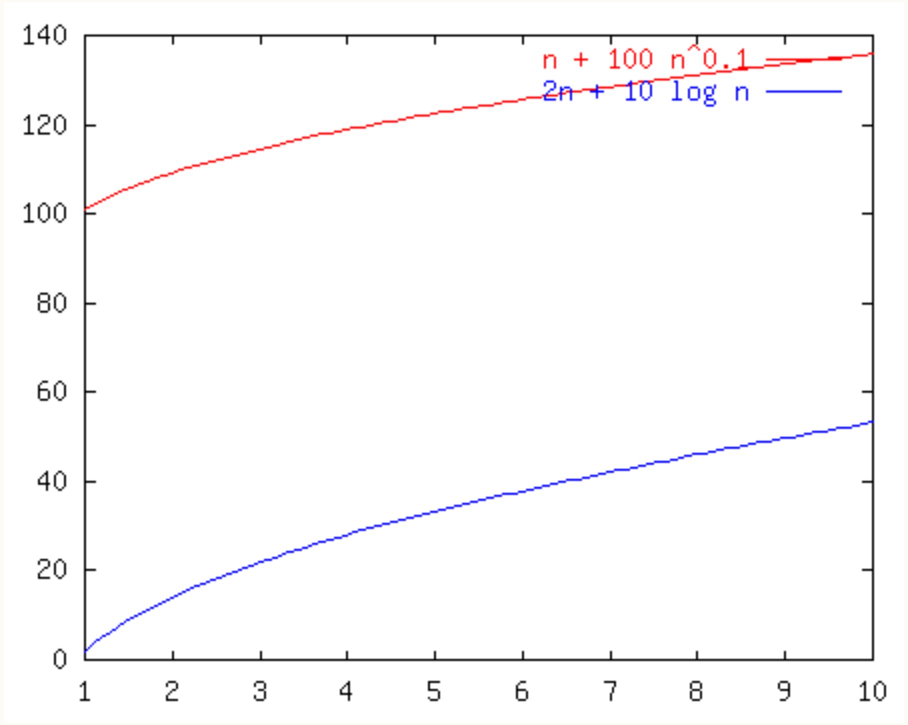
$n + 100n^{0.1}$ vs. $2n + 10 \log n$

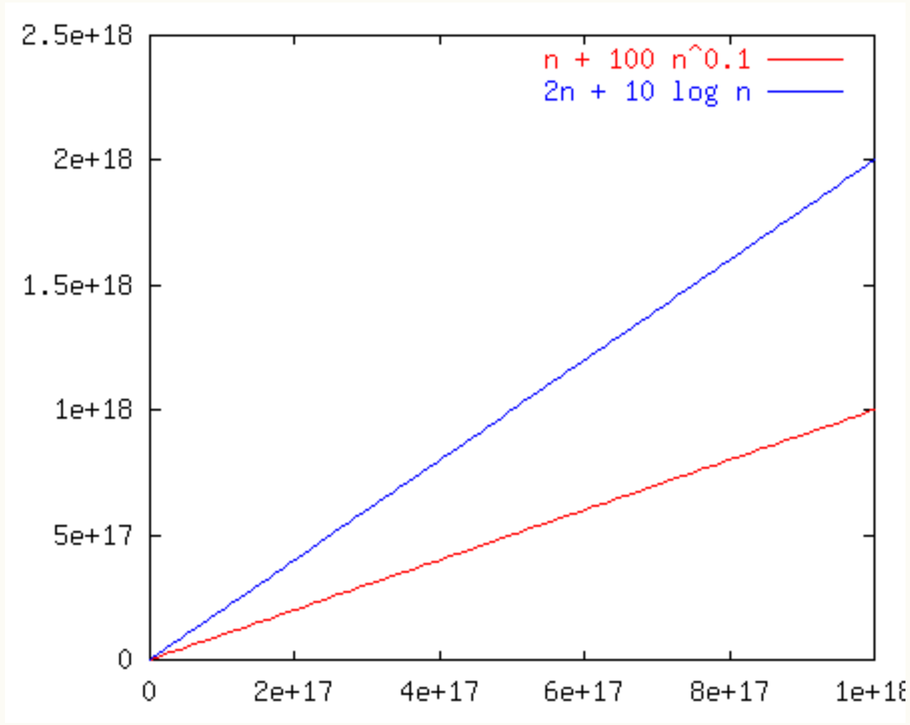


Race III

a. Left
b. Right
c. Tied
d. It depends

$n + 100n^{0.1}$ vs. **$2n + 10 \log n$**



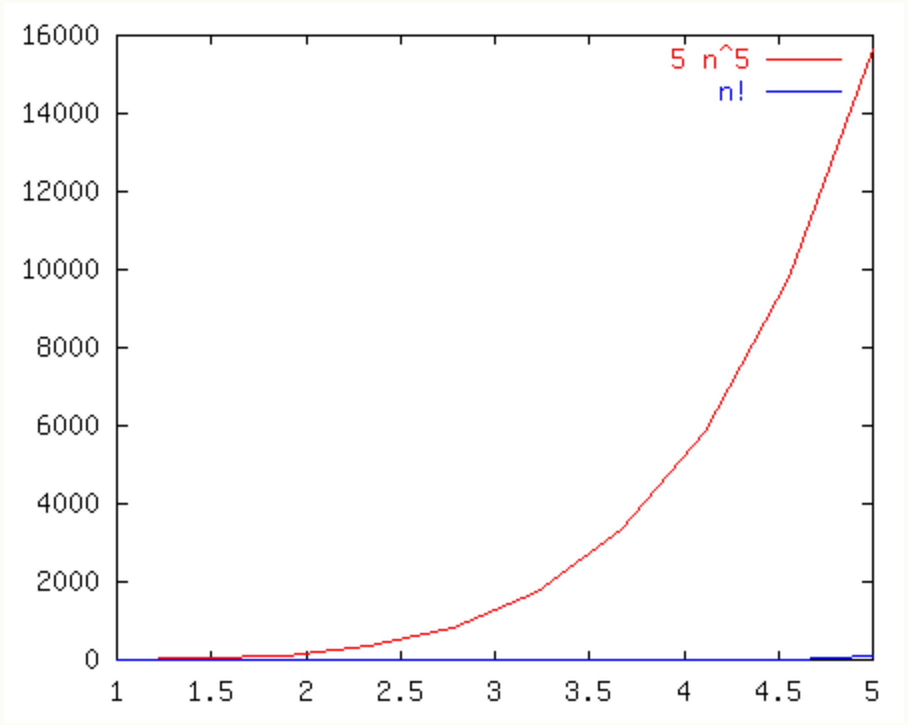


Although left seems faster, asymptotically it is a TIE

Race IV

a. *Left*
b. *Right*
c. *Tied*
d. *It depends*

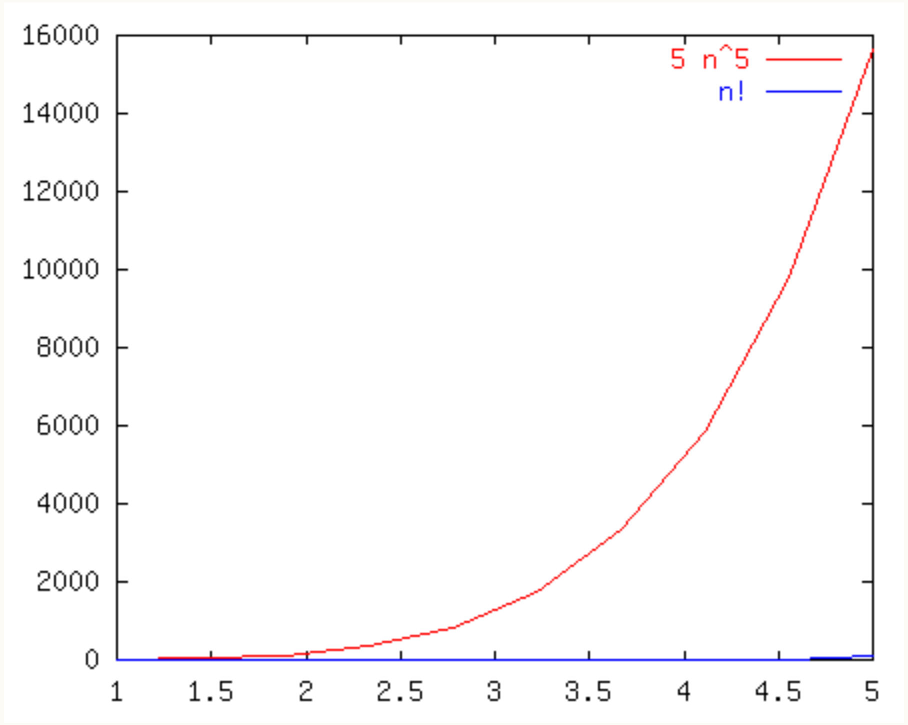
$5n^5$ *vs.* $n!$

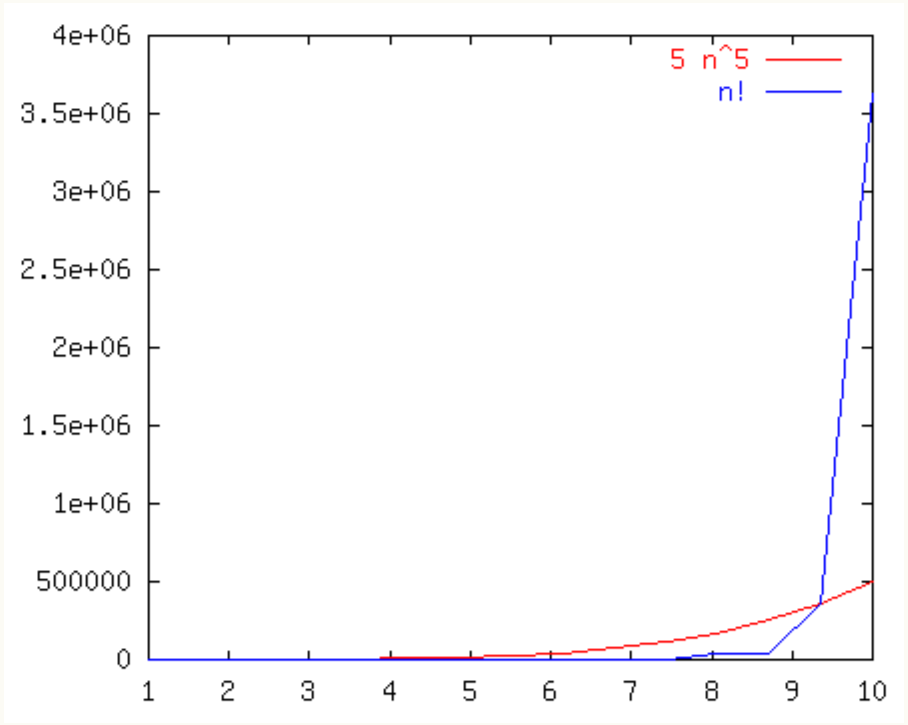


Race IV

a. *Left*
b. *Right*
c. *Tied*
d. *It depends*

$5n^5$ *vs.* $n!$

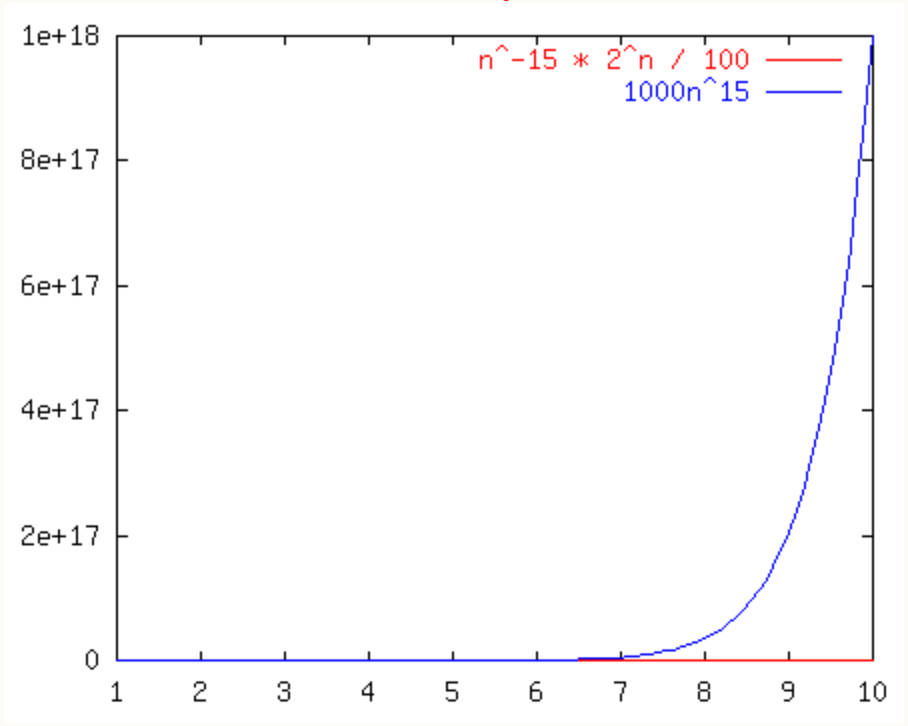




Race V

a. *Left*
b. *Right*
c. *Tied*
d. *It depends*

$n^{-15}2^n/100$ *vs.* $1000n^{15}$



Race V

a. Left
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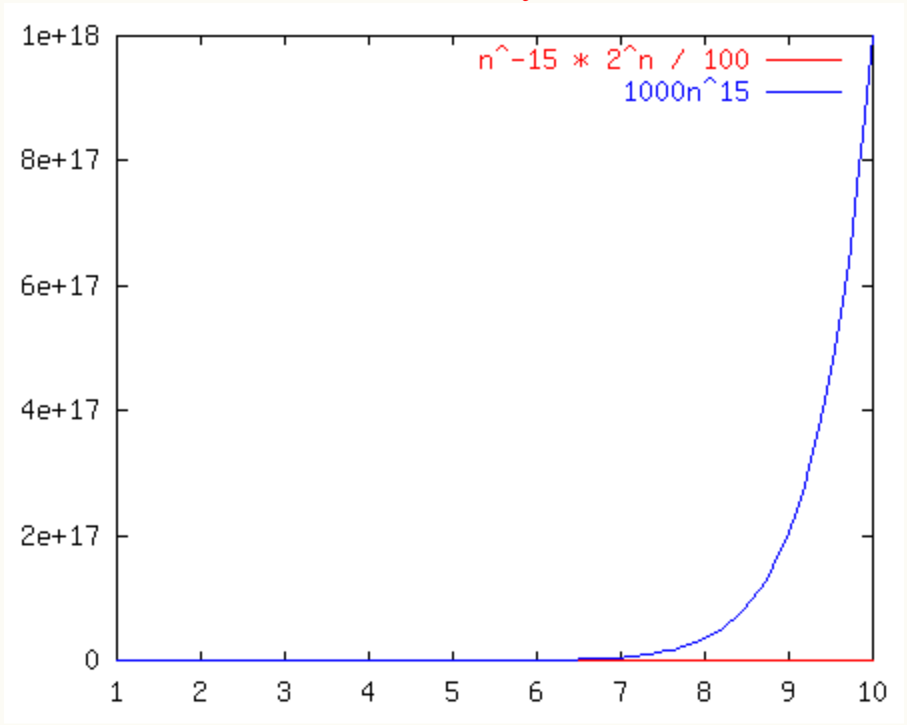
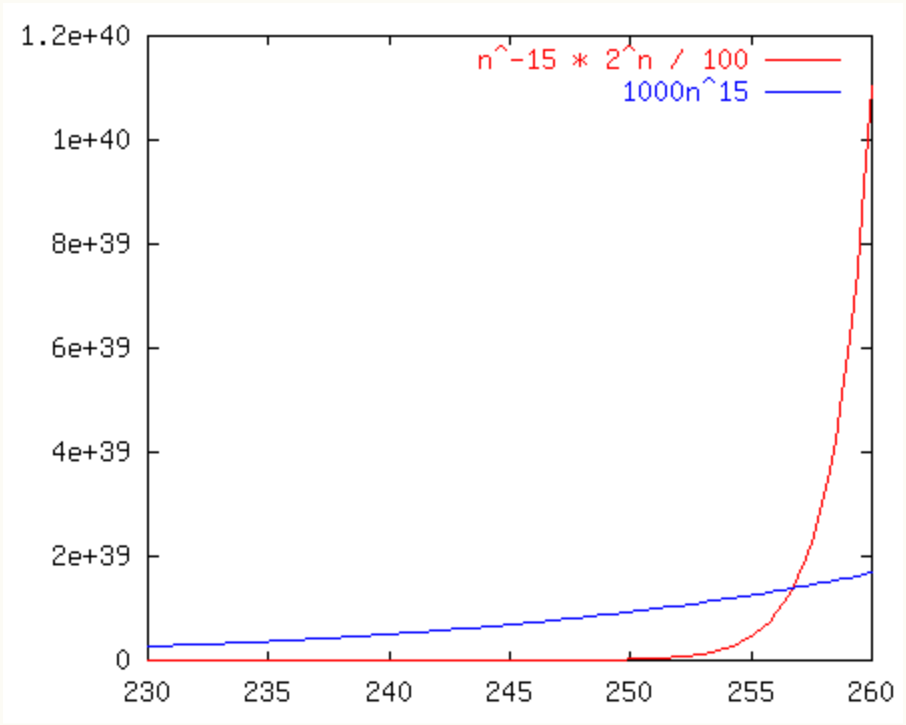
d. It depends

$n^{-15} 2^n/100$ *vs.* $1000n^{15}$



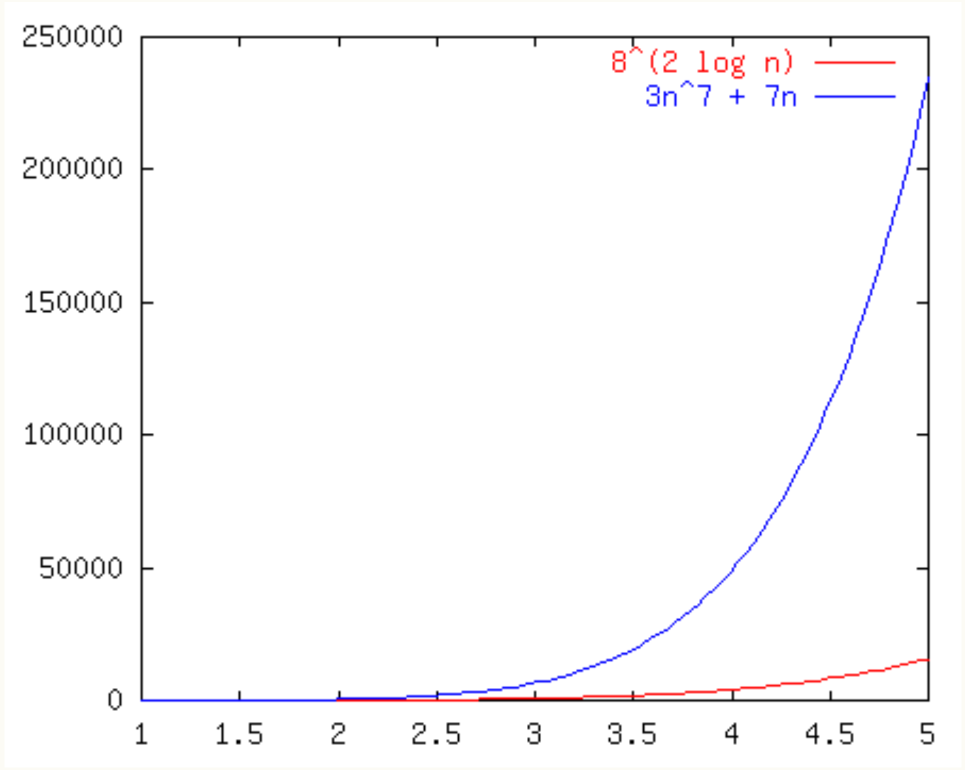


Any exponential is slower than any polynomial.
It doesn't even take that long here (~250 input size)

Race VI

a. Left
b. Right
c. Tied
d. It depends

$8^{2\log_2(n)}$ *vs.* $3n^7 + 7n$



Log Rules:

1) $\log(mn) = \log(m) + \log(n)$
2) $\log(m/n) = \log(m) - \log(n)$
3) $\log(m^n) = n \cdot \log(m)$
4) $n = 2^k \rightarrow \log_2 n = k$

Log Rules:

1) log(mn) = log(m) + log(n)

2) log(m/n) = log(m) – log(n)

3) log(mⁿ) = n · log(m)

4) n = 2ᵏ → log n = k

Race VI

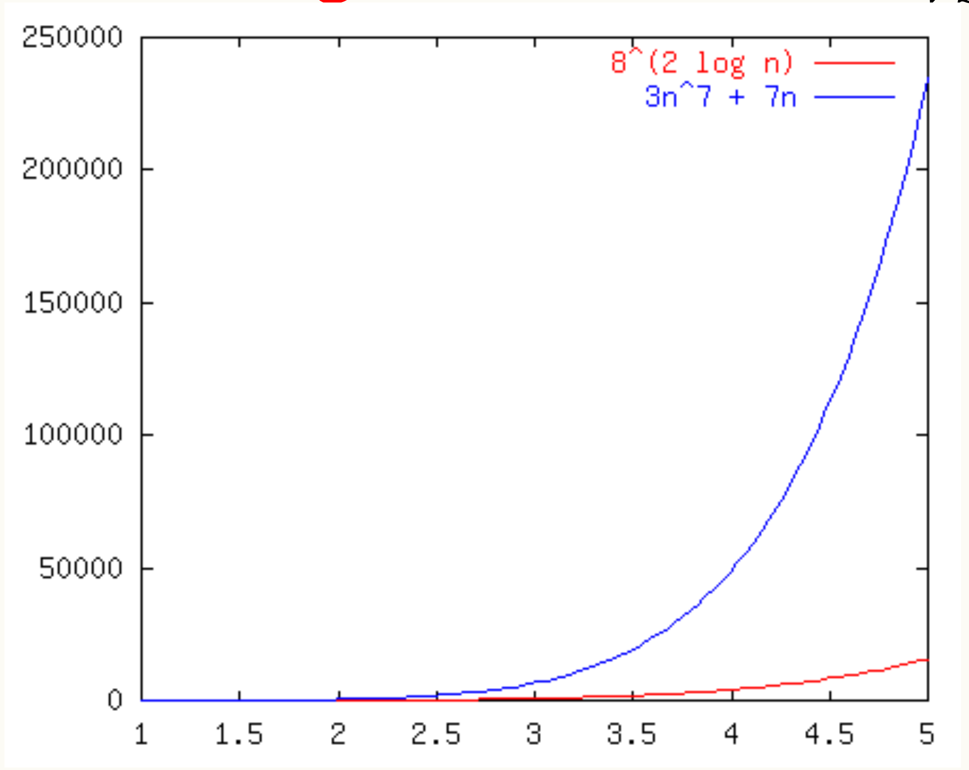
a. Left

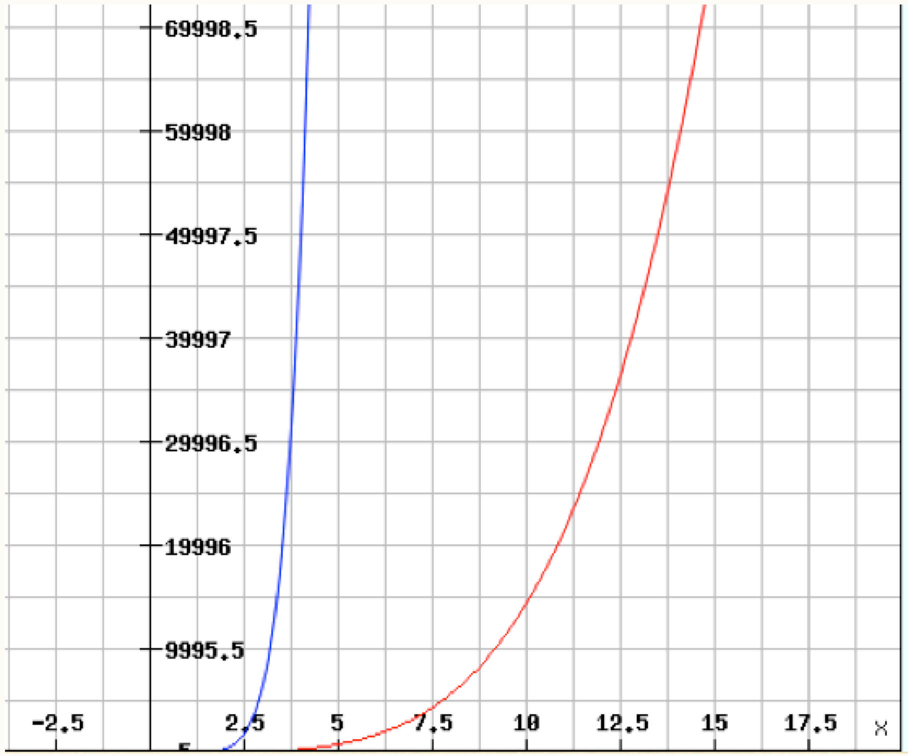
b. Right

c. Tied

d. It depends

$8^2 \log_2(n)$ *vs.* $3n^7 + 7n$





$$8^{2\lg(n)} = 8^{\lg(n^2)} = (2^3)^{\lg(n^2)} = 2^{3\lg(n^2)} = 2^{\lg(n^6)} = n^6$$

Log Aside

$\mathbf{log_a b}$ means "the exponent that turns **a** into **b**"

lg x means "$\mathbf{log_2 x}$" (the usual log in CS)

log x means "$\mathbf{log_{10} x}$" (the common log)

ln x means "$\mathbf{log_e x}$" (the natural log)

- There's just a constant factor between the three main log bases, and asymptotically they behave equivalently.

Race VII

a. Left
b. Right
c. Tied
d. It depends

mn^3 *vs.* $2^m n$

Race VII

a. Left
b. Right
c. Tied
d. It depends

mn^3 vs. $2^m n$

It depends on values of m and n

Silicon Downs

Post #1	*Post #2*	*Winner*
$n^3 + 2n^2$	$100n^2 + 1000$	$O(n^2)$
$n^{0.1}$	$\log n$	$O(\log n)$
$n + 100n^{0.1}$	$2n + 10 \log n$	***TIE*** $O(n)$
$5n^5$	$n!$	$O(n^5)$
$n^{-15}2^n/100$	$1000n^{15}$	$O(n^{15})$
$8^{2\lg n}$	$3n^7 + 7n$	$O(n^6)$
mn^3	$2^m n$	***IT DEPENDS***

The fix sheet and what we care about most

- The fix sheet (typical growth rates in order)

– constant:	$O(1)$	
– logarithmic:	$O(\log n)$	$(\log_k n, \log n^2 \in O(\log n))$
– poly-log:	$O(\log^k n)$	(k is a constant >1)
– Sub-linear:	$O(n^c)$	(c is a constant, $0 < c < 1$)
– linear:	$O(n)$	
– (log-linear):	$O(n \log n)$	(usually called "n log n")
– (superlinear):	$O(n^{1+c})$	(c is a constant, $0 < c < 1$)
– quadratic:	$O(n^2)$	
– cubic:	$O(n^3)$	
– polynomial:	$O(n^k)$	(k is a constant)
– exponential:	$O(c^n)$	(c is a constant > 1)

Tractable

Intractable!

Name-drop your friends

– constant:	$O(1)$
– Logarithmic:	$O(\log n)$
– poly-log:	$O(\log^k n)$
– Sub-linear:	$O(n^c)$
– linear:	$O(n)$
– (log-linear):	$O(n \log n)$
– (superlinear):	$O(n^{1+c})$
– quadratic:	$O(n^2)$
– cubic:	$O(n^3)$
– polynomial:	$O(n^k)$
– exponential:	$O(c^n)$

Casually name-drop the appropriate terms in order to sound bracingly cool to colleagues: "Oh, linear search? I hear it's sub-linear on quantum computers, though. Wild, eh?"

USE those cheat sheets!

- Which is faster, n^3 or $n^3 \log n$?

$$n^3 * 1 \qquad \text{vs. } n^3 * \log n$$

- Which is faster, n^3 or $n^{3.01}/\log n$?

(Split it up and use the "dominance" relationships.)

$$n^3 * 1 \qquad \text{vs. } n^3 * \; n^{0.01} / \log n$$

Clicker Question

Which of the following functions is likely to grow the fastest, meaning that the algorithm is likely to take the most steps, as the input size, n, grows sufficiently large?

A. O(n)
B. O(sqrt (n))
C. O (log n)
D. O (n log n)
E. They would all be about the same.

Clicker Question (answer)

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B. O(sqrt (n))
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D. O (n log n)
E. They would all be about the same.

Clicker Question

Suppose we have 4 programs, A-D, that run algorithms of the time complexities given. Which program will finish first, when executing the programs on input size n=10?

A. O(n)
B. O(sqrt (n))
C. O (log n)
D. O (n log n)
E. Impossible to tell

Clicker Question (Answer)

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A. O(n)
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C. O (log n)
D. O (n log n)
E. Impossible to tell

Clicker Question

Which of the following statements about complexity is true? Choose the best answer

A. The set of functions in O(n^4) have a fairly slow growth rate

B. O(n) doesn't grow very quickly

C. Big-O functions with the fastest growth rate represent the fastest algorithms, most of the time

D. Asymptotic complexity deals with relatively small input sizes

Clicker Question (answer)

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C. Big-O functions with the fastest growth rate represent the fastest algorithms, most of the time

D. Asymptotic complexity deals with relatively small input sizes

Order Notation – Big-O

- $T(n) \in O(f(n))$ if there are constants $c > 0$ and n_0 such that $T(n) \leq c\, f(n)$ for all $n \geq n_0$

- Why the $\in$?

  (Many people write $T(n)=O(f(n))$,
  but this is sloppy. The $\in$ shows you why
  you should never write $O(f(n))=T(n)$,
  with the big-O on the left-hand side.)

How to formalize winning?

- How to formally say that there’s some crossover point, after which one function is bigger than the other?
- How to formally say that you don’t care about a constant factor between the two functions?

Order Notation – Big-O

- $T(n) \in O(f(n))$ if there are constants $c > 0$ and n_0 such that $T(n) \leq c\, f(n)$ for all $n \geq n_0$

- Intuitively, what does this all mean?

The function f(n) is sort of, asymptotically "greater than or equal to" the function T(n).

In the "long run", f(n) (multiplied by a suitable constant) will upper-bound T(n).

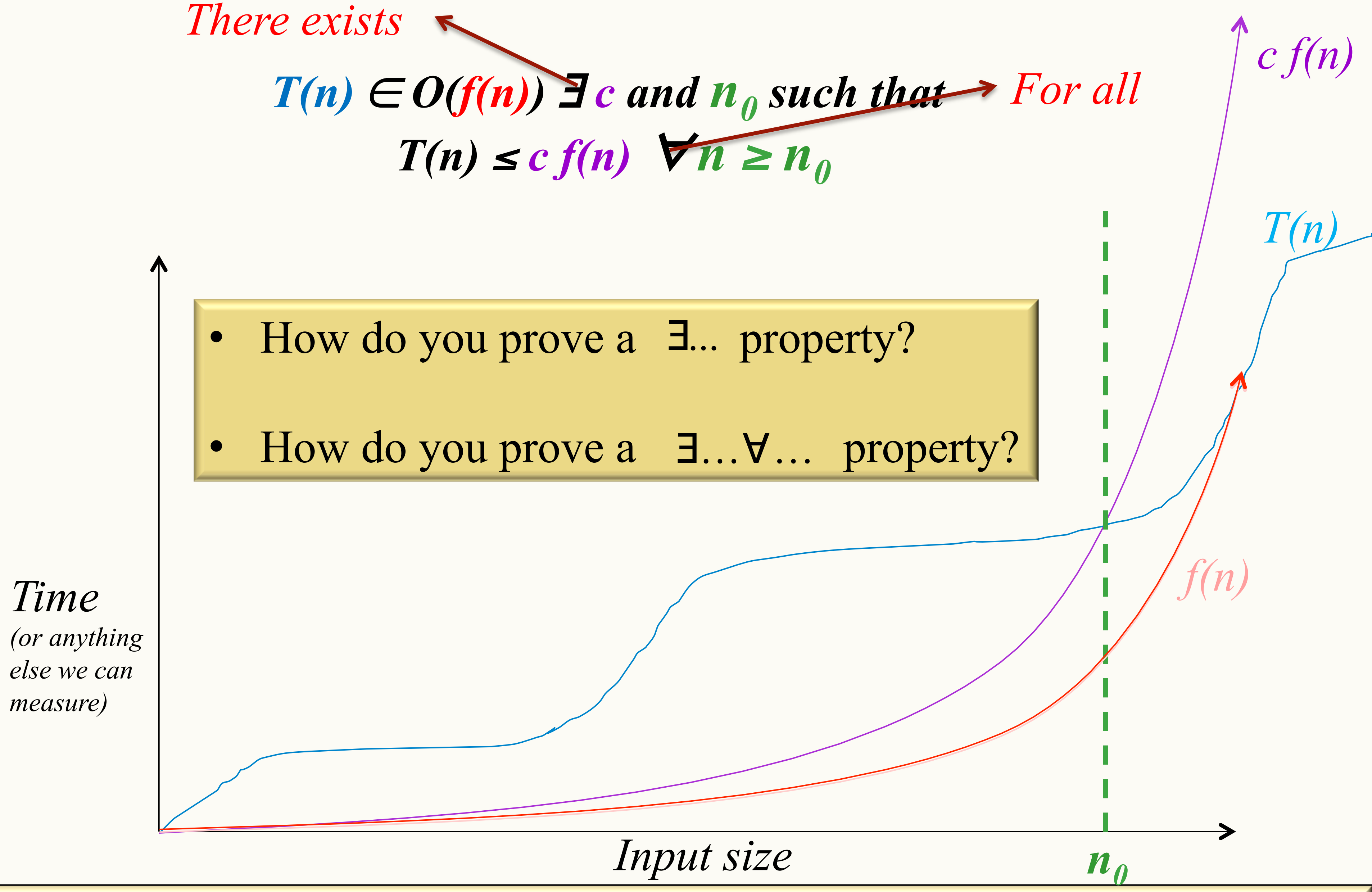
Proving a Big-O

There exists

$T(n) \in O(f(n))\ \exists\, c$ and n_0 such that

For all

$T(n) \le c\, f(n)\ \ \forall n \ge n_0$



- How do you prove a ∃... property?
- How do you prove a ∃…∀… property?

Proving a "There exists" Property

How do you prove "There exists a good restaurant in Vancouver"?

How do you prove a property like

$$\boxed{\exists c\left[c = 3c + 1\right]}$$

Proving a ∃…∀… Property

How do you prove "There exists a restaurant in Vancouver, where all items on the menu are less than $10"?

How do you prove a property like

$$\boxed{\exists c \forall x \left\lfloor c \le x^2 - 10 \right\rfloor}$$

Proving a Big-O

Formally, to prove $T(n) \in O(f(n))$, you must show:

$$\boxed{\exists c > 0, n_0 \forall n > n_0 \left[T(n) \le cf(n) \right]}$$

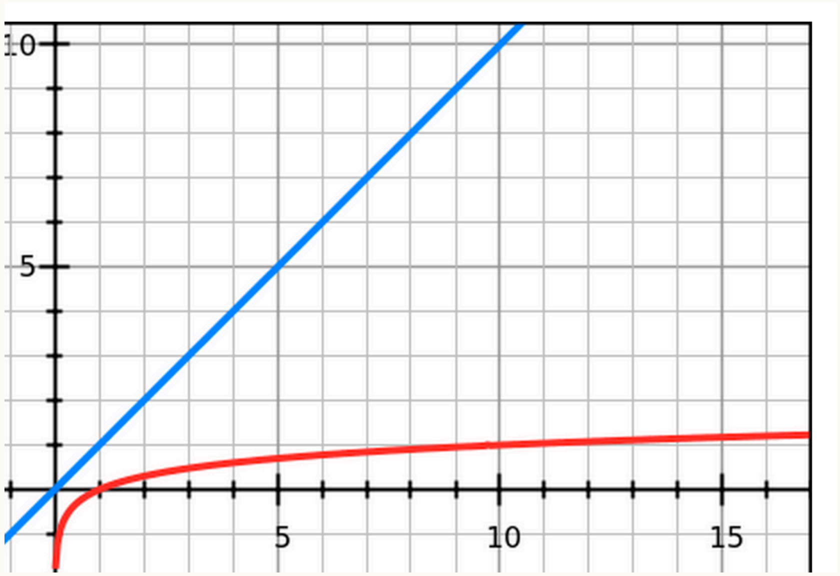
So, we have to come up with specific values of c and n_0 that "work", where "work" means that for any $n > n_0$ that someone picks, the formula holds:

$$\boxed{\left[T(n) \le cf(n) \right]}$$

Prove $n \log n \in O(n^2)$

- Guess or figure out values of c and n_0 that will work.

$$n \log n \le cn^2$$
$$\log n \le cn$$



- This is fairly trivial: log n <= n (for n>1)

c=1 and n_0 = 1 works!

Aside: Writing Proofs

- In lecture, my goal is to give you intuition.
  - I will just sketch the main points, but not fill in all details.
- When you ***write*** a proof (homework, exam, reports, papers), be sure to write it out formally!
  - Standard format makes it much easier to write!
    - Class website has links to notes with standard tricks, examples
    - Textbook has good examples of proofs, too.
    - Copy the style, structure, and format of these proofs.
  - On exams and homeworks, you'll get more credit.
  - In real life, people will believe you more.

To Prove $n \log n \in O(n^2)$

Proof:

By the definition of big-O, we must find values of c and n_0 such that for all $n \geq n_0$, $n \log n \leq cn^2$.

Consider c=1 and $n_0 = 1$.

For all $n \geq 1$, $\log n \leq n$.

Therefore, $\log n \leq cn$, since c=1.

Multiplying both sides by n (and since $n \geq n_0 = 1$), we have

$$n \log n \leq cn^2.$$

Therefore, $n \log n \in O(n^2)$.

(This is more detail than you'll use in the future, but until you learn what you can skip, fill in the details.)

Example

- Prove $T(n) = n^3 + 20n + 1 \in O(n^3)$
  - $n^3 + 20n + 1 \leq cn^3$ for $n > n_0$
  - $1 + 20/n^2 + 1/n^3 \leq c \rightarrow$ holds for $c=22$ and $n_0 = 1$
- Prove $T(n) = n^3 + 20n + 1 \in O(n^4)$
  - $n^3 + 20n + 1 \leq cn^4$ for $n > n_0$
  - $1/n + 20/n^3 + 1/n^4 \leq c \rightarrow$ holds for $c=22$ and $n_0 = 1$
- Prove $T(n) = n^3 + 20n + 1 \notin O(n^2)$
  - $n^3 + 20n + 1 \leq cn^2$ for $n > n_0$
  - $n + 20/n + 1/n^2 \leq c \rightarrow$ You cannot find such c or n_0

Computing Big-O

- If $T(n)$ is a polynomial of degree d
  - (i.e., $T(n) = a_0 + a_1 n + a_2 n^2 + \ldots + a_d n^d$),
- then its Big-O estimate is simply the largest term without its coefficient, that is, $T(n) \in O(n^d)$.

- If $T_1(n) \in O(f(n))$ and $T_2(n) \in O(g(n))$, then
  - $T_1(n) + T_2(n) \in O(\max(f(n), g(n)))$.
  - $T_1(n) = 4 n^{3/2} + 9$
  - $T_2(n) = 30 n \lg n + 17n$
  - $T(n) = T_1(n) + T_2(n) \in O(\max(n^{3/2}, n \log n) = O(n^{3/2})$

More Example

- Compute Big-O with witnesses *c* and n_0 for
  - $T(n) = 25n^2 - 50n + 110$.

$25n^2 - 50n + 110 \le 25n^2 + 50n + 110 \le cn^2$

$25 + 50/n + 110/n^2 \le c$

$T(n) \in O(n^2) \quad c=27,\ n_0=110$

Triangle inequality

$|a+b| \le |a| + |b|$

(substitute –b with b)

$|a-b| \le |a| + |-b| \le |a| + |b|$

We are interested in the "tightest" Big-O estimate and not necessarily the smallest c and n_0

More Example

- **Example** Compute Big-O with witnesses *c* and n_0 for $T(n) = 10^6$.

$10^6 \leq c$

$T(n) \in O(1)$ $c=10^6$, n_0 = *whatever*

- **Example** Compute Big-O with witnesses *c* and n_0 for $T(n) = \log(n!)$.

$\log(n!) = \log(1*2..*n)$

$= \log(1) + \log(2) + \ldots + \log(n)$

$\leq \log(n) + \log(n) + \ldots + \log(n)$

$\leq n \log(n) <= cn \log(n)$

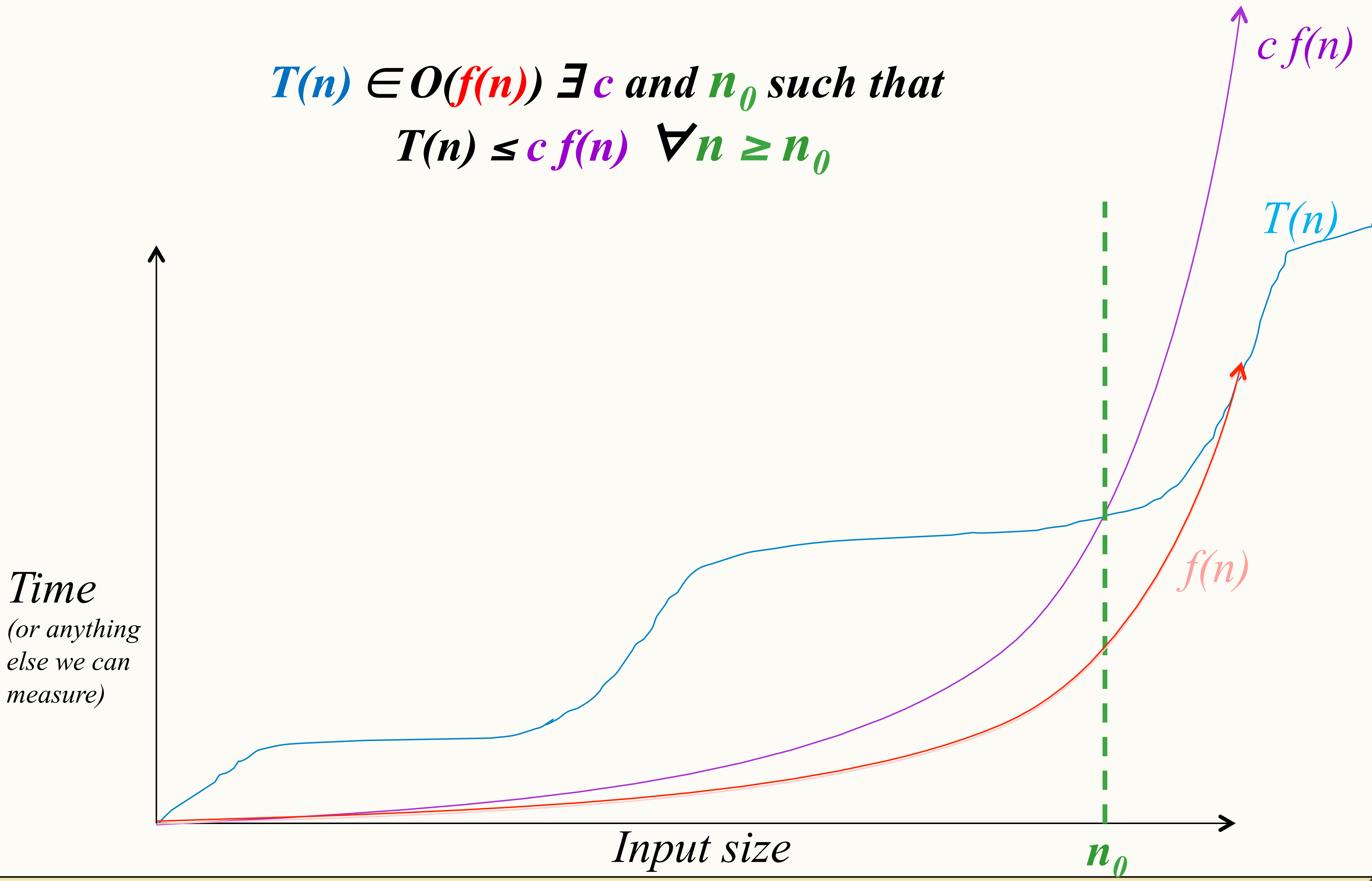
Log Rules:

1) $\log(mn) = \log(m) + \log(n)$
2) $\log(m/n) = \log(m) - \log(n)$
3) $\log(m^n) = n \cdot \log(m)$

$T(n) \in O(n \log(n))$ $c=10$, $n_0 = 10$

Proving a Big-O

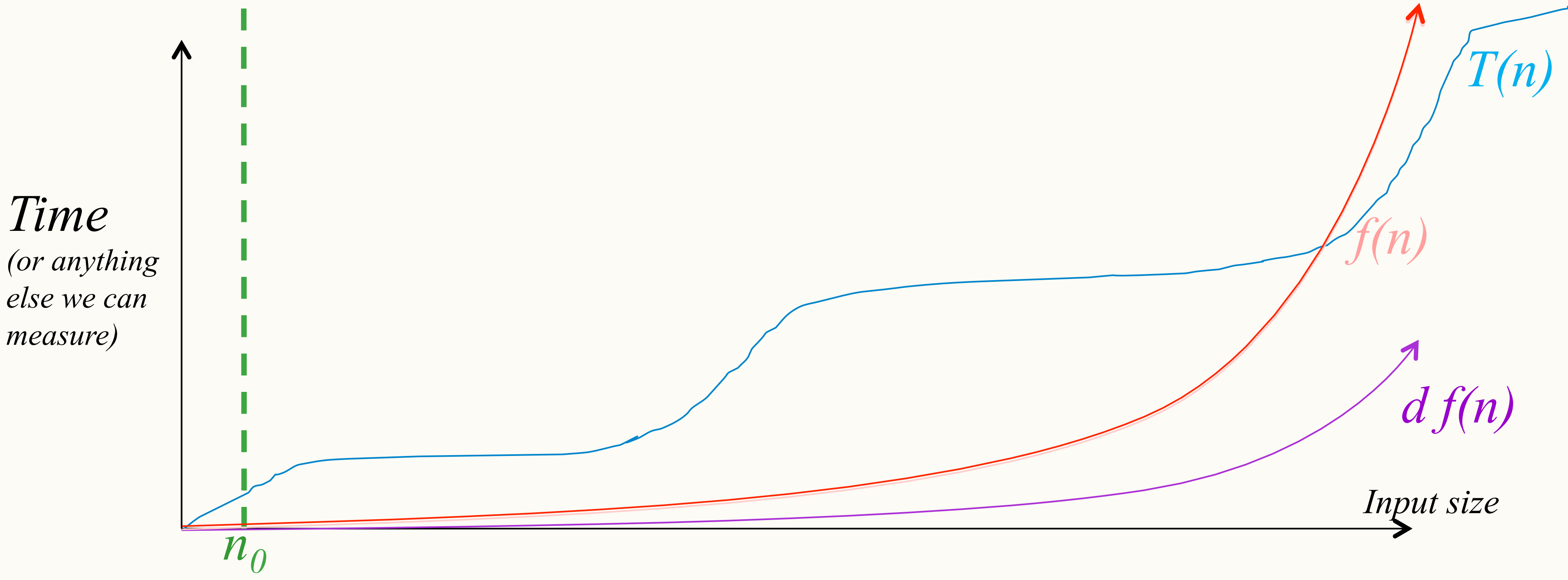
$$T(n) \in O(f(n))\ \exists\, c \text{ and } n_0 \text{ such that}$$
$$T(n) \le c\, f(n)\quad \forall n \ge n_0$$



Big-Omega (Ω) notation

- Just as Big-O provides an *upper* bound, there exists Big-Omega (Ω) notation to estimate the *lower* bound of an algorithm, meaning that, in the worst case, the algorithm takes at least so many steps:

$$T(n) \in \Omega(f(n)) \text{ if } \exists\, d \text{ and } n_0 \text{ such that}$$
$$d\, f(n) \leq T(n) \;\; \forall\, n \geq n_0$$

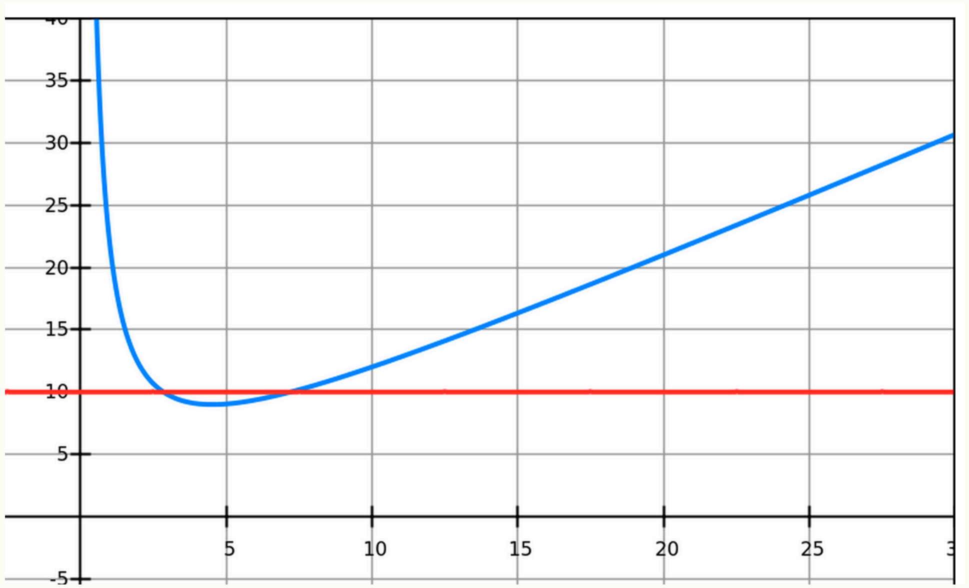


Proving Big-Ω

- Just like proving Big-O, but backwards…

- Prove $T(n) = n^3 + 20n + 1 \in \Omega(n^2)$

$$dn^2 \le n^3 + 20n + 1$$
$$d \le n + 20/n + 1/n^2$$
$$d = 10, \quad n_0 = 20$$

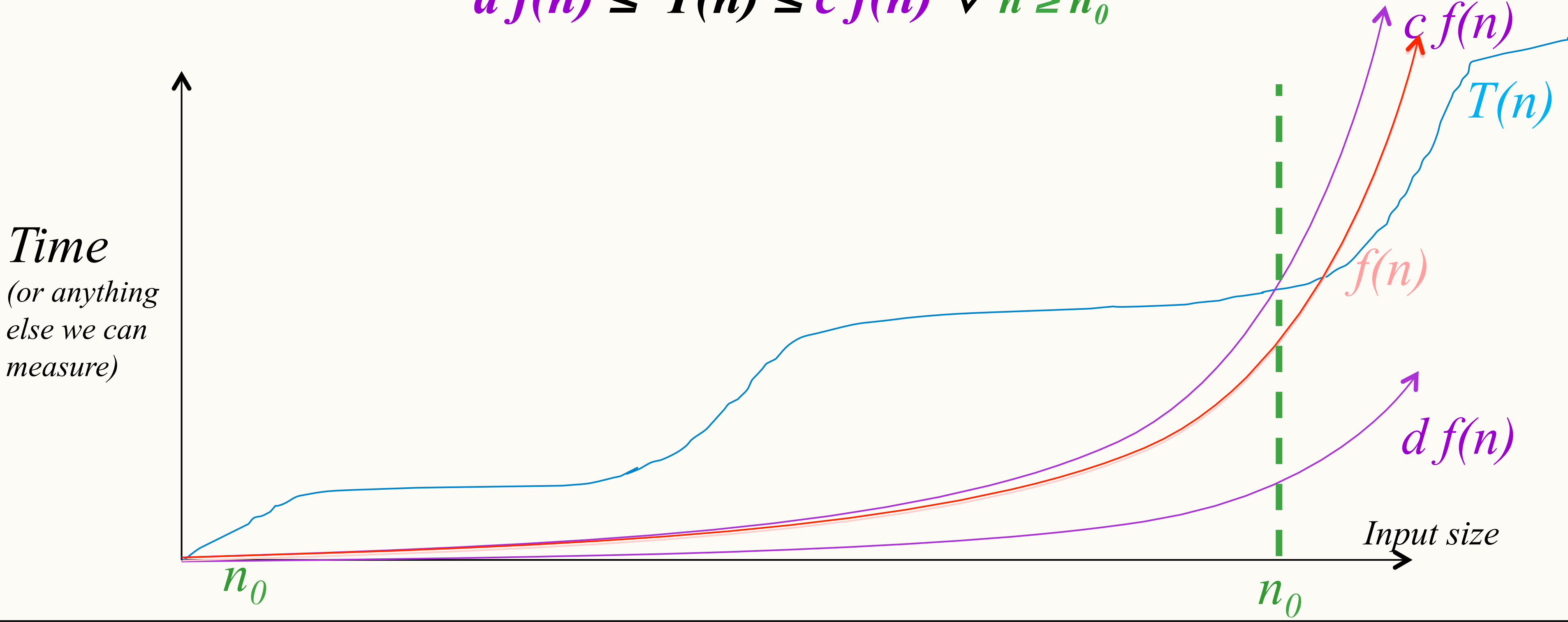


Big-Theta (Θ) notation

- Furthermore, each algorithm has both an upper bound and a lower bound, and when these correspond to the same growth order function, the result is called Big-Theta (Θ) notation.

$$T(n) \in \Theta(f(n)) \text{ if } \exists\, c,\, d \text{ and } n_0 \text{ such that}$$

$$d\, f(n) \leq T(n) \leq c\, f(n) \;\; \forall\, n \geq n_0$$



Examples

$10{,}000\, n^2 + 25\, n \in \Theta(n^2)$

$10^{-10}\, n^2 \in \Theta(n^2)$

$n \log n \in O(n^2)$

$n \log n \in \Omega(n)$

$n^3 + 4 \in O(n^4)$ but not $\Theta(n^4)$

$n^3 + 4 \in \Omega(n^2)$ but not $\Theta(n^2)$

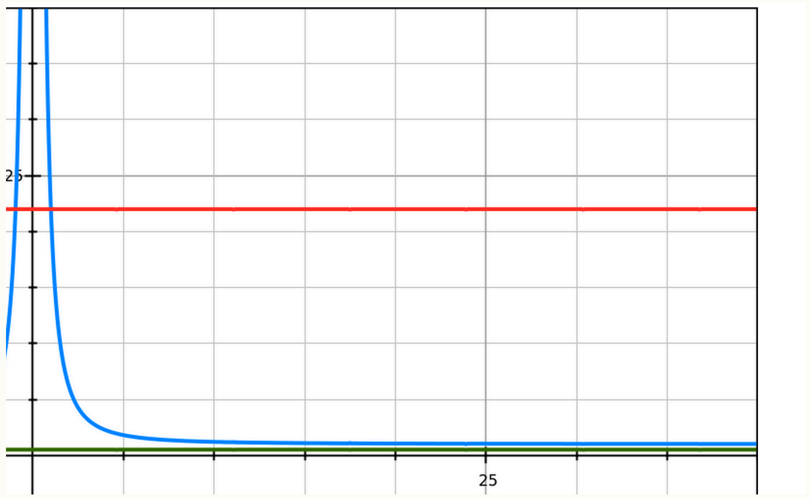
Proving Big-Θ

- Just prove Big-O and Big-Ω
- Prove $T(n) = n^3 + 20n + 1 \in \Theta(n^3)$

$dn^3 \leq n^3 + 20n + 1 \leq cn^3$ for $n > n_0$

$d \leq 1 + 20/n^2 + 1/n^3 \leq c$

holds for $d=1$, $c=22$, $n_0 = 105$

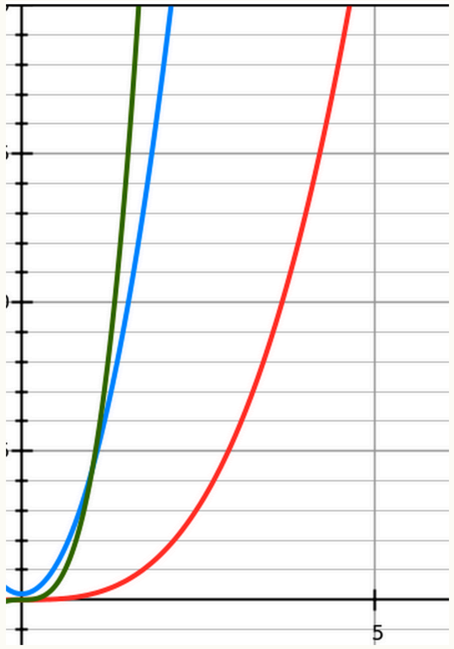


Proving Big-Θ

- Just prove Big-O and Big-Ω
- Prove $T(n) = n^3 + 20n + 1 \in \Theta(n^3)$

$dn^3 \le n^3 + 20n + 1 \le cn^3$ for $n > n_0$

holds for $d=1$, $c=22$, $n_0 = 105$



Types of analysis

- Bound flavor
  - upper bound (O)
  - lower bound (Ω)
  - asymptotically tight (Θ)

- Analysis case
  - worst case (adversary)
  - average case
  - best case
  - "common" case

- Analysis quality
  - loose bound (any true analysis)
  - tight bound

“Tight” Bound

There are at least three common usages for calling a bound “tight”:

1. Big-Theta, “asymptotically tight”
2. “no better bound which is asymptotically different”
3. Big-O upper bound on run time of an algorithm matches provable worst-case lower-bound on any solution algorithm.

"Tight" Bound – Def. 1

1. Big-Theta, "asymptotically tight"

This definition is formal and clear:

$T(n) \in \Theta(f(n))$ if $T(n) \in O(f(n))$ and $T(n) \in \Omega(f(n))$

but it is too rigid to capture practical intuition.

For example, what if

$T(n) =$ `(n%2==0)? n*n : 1`

Is $T(n) \in O(n^2)$?

Is $T(n) \in \Theta(n^2)$?

“Tight” Bound – Def. 2

- “no better bound which is asymptotically different”
- This is the most common definition, and captures what people usually want to say, but it’s not formal.
  - E.g., given same $T(n)$, we want $T(n) \in O(n^2)$ to be considered “tight”, but not $T(n) \in O(n^3)$
  - But, $T(n)$ is NOT $\Theta(n^2)$, so isn’t $T(n) \in O(T(n))$ a tighter bound?

“Tight” Bound – Def. 2

- “no better ‘reasonable’ bound which is asymptotically different”
- This is the most common definition, and captures what people usually want to say, but it’s not formal.

“Reasonable” is defined subjectively, but it basically means a simple combination of normal, common functions, i.e., anything on our list of common asymptotic complexity categories (e.g., log n, n, n^k, 2^n, n!, etc.). There should be no lower-order terms, and no unnecessary coefficients.

This is the definition we’ll use in CPSC 221 unless stated otherwise.

“Tight” Bound – Def. 3

- Big-O upper bound on run time of an algorithm matches provable lower-bound on any algorithm.
- The definition used in more advanced, theoretical computer science:
  - Upper bound is on a specific algorithm.
  - Lower bound is on the problem in general.
  - If the two match, you can’t get an asymptotically better algorithm.
- This is beyond this course, for the most part.
  - (Examples: Searching and Sorting…)

Bounding Searching and Sorting

- Searching an unsorted list using comparisons takes $\Omega(n)$ time (lower bound)
  - Linear search takes $O(n)$ time (matching upper bound)
- Sorting a list using comparisons takes $\Omega(n \lg n)$ time (lower bound)
  - Mergesort takes $O(n \lg n)$ time (matching upper bound)
  - More on this later!

Analyzing Code

- But how can we obtain T(n) from an algorithm/ code

• C++ operations	- constant time
• consecutive stmts	- sum of times
• conditionals	- max of branches, condition
• loops	- sum of iterations
• function calls	- cost of function body

Analyzing Code

```
find(key, array)
    for i = 1 to length(array) do
        if array[i] == key
            return i

    return -1
```

- Step 1: What's the input size **n**?
- Step 2: What kind of analysis should we perform?
  - Worst-case? Best-case? Average-case?
- Step 3: How much does each line cost? (Are lines the right unit?)

Analyzing Code

```
find(key, array)
    for i = 1 to length(array) do
        if array[i] == key
            return i

    return -1
```

- Step 4: What's **`T(n)`** in its raw form?
- Step 5: Simplify **`T(n)`** and convert to order notation. (Also, which order notation: O, Θ, Ω?)
- Step 6: **Prove** the asymptotic bound by finding constants **c** and $\mathbf{n_0}$ such that
  - for all $\mathbf{n \geq n_0}$, $\mathbf{T(n) \leq cn}$.

Example 1

```
for i = 1 to n do          1                ⎤ n times
    for j = 1 to n do      1   ⎤ n times    ⎥
        sum = sum + 1      1   ⎦            ⎦
```

- This example is pretty straightforward. Each loop goes *n* times, and a constant amount of work is done on the inside.

$$T(n) = \sum_{i=1}^{n}(1 + \sum_{j=1}^{n} 2) = \sum_{i=1}^{n}(1 + 2n) = n + 2n^2 = O(n^2)$$

Example 1 (simpler version)

```
for i = 1 to n do                 1   ]  n times
    for j = 1 to n do             1   ]
        sum = sum + 1             1   ]n times
```

- Count the number of times `sum = sum + 1` occurs

$$T(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} 1 = \sum_{i=1}^{n} n = n^2 = O(n^2)$$

Example 2

```
i = 1
    while i < n do
        for j = i to n do
            sum = sum + 1
        i++
```

Time complexity:

a. O(n)
b. O(n lg n)
c. $O(n^2)$
d. $O(n^2 \lg n)$
e. None of these

Example 2 (Pure Math Approach)

```
i = 1                        takes "1" step
while i < n do               i varies 1 to n-1
  for j = i to n do          j varies i to n
    sum = sum + 1            takes "1" step
  i++                        takes "1" step
```

Now, we write a function T(n) that adds all of these up, summing over the iterations of the two loops:

$$T(n) = 1 + \sum_{i=1}^{n-1}\left(1 + \sum_{j=i}^{n} 1\right)$$

Example 2 (Pure Math Approach)

Here's our function for the runtime of the code:

$$T(n) = 1 + \sum_{i=1}^{n-1}\left(1 + \sum_{j=i}^{n} 1\right)$$

Summing 1 for *j* from *i* to *n* is just going to be 1 added together (*n*-*i*+1) times, which is (*n*-*i*+1):

$$T(n) = 1 + \sum_{i=1}^{n-1}(1 + n - i + 1) = 1 + \sum_{i=1}^{n-1}(n - i + 2)$$

Example 2 (Pure Math Approach)

Here's our function for the runtime of the code:

$$T(n) = 1 + \sum_{i=1}^{n-1}(1 + n - i + 1) = 1 + \sum_{i=1}^{n-1}(n - i + 2)$$

The *n* and 2 terms don't change as *i* changes. So, we can pull them out (and multiply by the number of times they're added):

$$T(n) = 1 + n(n-1) + 2(n-1) - \sum_{i=1}^{n-1} i$$

And, we know that $\sum_{i=1}^{k} i = k(k+1)/2$, so:

$$T(n) = 1 + n^2 - n + 2n - 2 - \frac{(n-1)n}{2}$$

Example 2 (Pure Math Approach)

Here's our function for the runtime of the code:

$$T(n) = 1 + n^2 - n + 2n - 2 - \frac{(n-1)n}{2}$$

$$= n^2 + n - 1 - \frac{n^2}{2} + \frac{n}{2} = \frac{n^2}{2} + \frac{3n}{2} - 1$$

So, $T(n) = \frac{n^2}{2} + \frac{3n}{2} - 1$.

Drop low-order terms and the ½ coefficient, and we find:

$T(n) \in \Theta(n^2)$.

Yay!!!

Example 2 (Simplified Math Approach)

```
i = 1
   while i < n do
      for j = i to n do
         sum = sum + 1
      i++
```

Count this line

$$T(n) = \sum_{i=1}^{n-1} \sum_{j=i}^{n} 1$$

The second sigma is n-i+1

$$T(n) = \sum_{i=1}^{n-1} (n - i + 1) = n + n - 1 + \ldots + 2$$

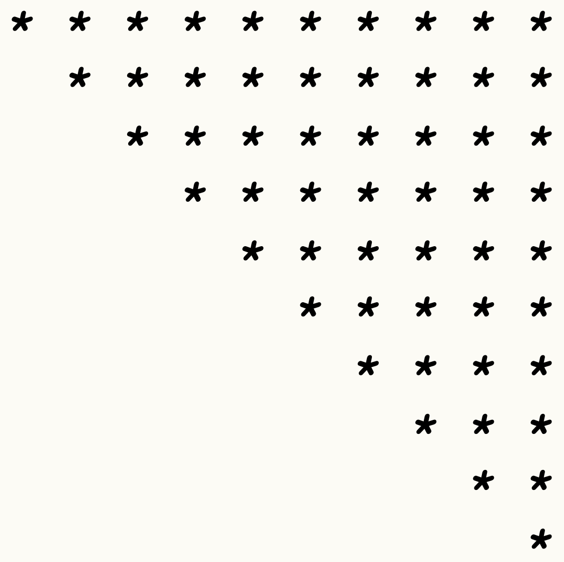
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$T(n) = n(n+1)/2 \in \Theta(n^2)$$

Example 2 Pretty Pictures Approach

```
i = 1                         takes "1" step
while i < n do                i varies 1 to n-1
  for j = i to n do           j varies i to n
    sum = sum + 1             takes "1" step
  i++                         takes "1" step
```

- Imagine drawing one point for each time the "sum=sum+1" line gets executed. In the first iteration of the outer loop, you'd draw n points. In the second, n-1. Then n-2, n-3, and so on down to (about) 1. Let's draw that picture…



Example 2 Pretty Pictures Approach

```
          n columns
* * * * * * * * * *
  * * * * * * * * *
    * * * * * * * *
      * * * * * * *     n rows
        * * * * * *
          * * * * *
            * * * *
              * * *
                * *
                  *
```

- It is a triangle and its area is proportional to runtime

$$T(n) = \frac{Base \times Height}{2} = \frac{n^2}{2} \in \Theta(n^2)$$

Example 2 (Faster/Slower Code Approach)

```
i = 1                        takes "1" step
while i < n do               i varies 1 to n-1
  for j = i to n do          j varies i to n
    sum = sum + 1            takes "1" step
  i++                        takes "1" step
```

- This code is "too hard" to deal with. So, let's find just an upper bound.
  - In which case we get to change the code so in any way that makes it run no faster (even if it runs slower).
  - We'll let j go from 1 to n rather than i to n. Since i ≥ 1, this is no less work than the code was already doing…

Example 2 (Faster/Slower Code Approach)

```
i = 1                              takes "1" step
while i < n do                     i varies 1 to n-1
  for j = 1 to n do                j varies i to n
    sum = sum + 1                  takes "1" step
  i++                              takes "1" step
```

- But this is just an upper bound $O(n^2)$, since we made the code run slower.

```
* * * * * * *                  * * * * * *
* * * * * * *                    * * * * *
* * * * * * *                      * * * *
* * * * * * *                        * * *
* * * * * * *                          * *
* * * * * * *                            *
```

- Could it actually run faster?

Example 2 (Faster/Slower Code Approach)

```
i = 1                        takes "1" step
while i < n do               i varies 1 to n-1
  for j = i to n do          j varies i to n
    sum = sum + 1            takes "1" step
  i++                        takes "1" step
```

- Let's do a lower-bound, in which case we can make the code run faster if we want.
  - Let's make j start at n-1. Does the code run faster? Is that helpful?

Runs faster but you get Ω(n) which is not what we want

```
*                    * * * * * *
*                      * * * * *
*                        * * * *
*                          * * *
*                            * *
*                              *
```

Example 2 (Faster/Slower Code Approach)

```
i = 1                          takes "1" step
while i < n do                 i varies 1 to n-1
  for j = i to n do            j varies i to n
    sum = sum + 1              takes "1" step
  i++                          takes "1" step
```

- Let's do a lower-bound, in which case we can make the code run faster if we want.
  - Let's make j start at n/2. Does the code run faster? Is that helpful?

Hard to argue that it is faster. Every inner loop now runs n/2 times

```
* * *                   * * * * * *
* * *                     * * * * *
* * *                       * * * *
* * *                         * * *
* * *                           * *
* * *                             *
```

Example 2(Faster /Slower Code Approach)

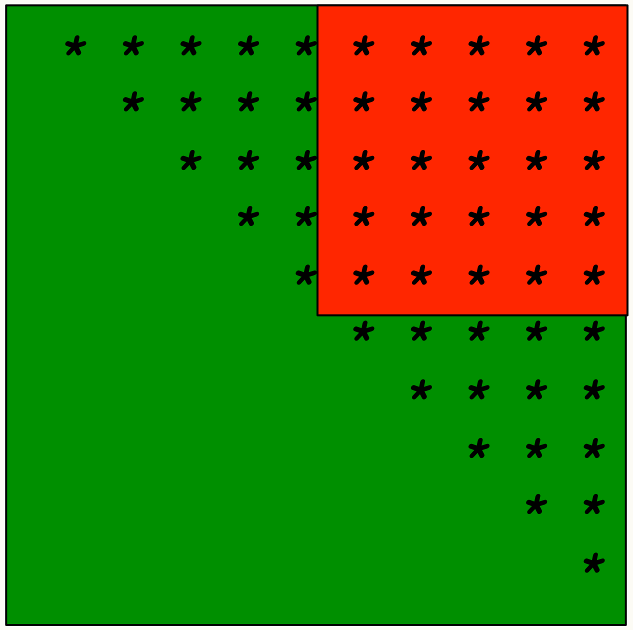
```
i = 1                           takes "1" step
while i < n/2 +1 do             goes n/2 times
   for j = n/2 +1 to n do       goes n/2 times
      sum = sum + 1             takes "1" step
   i++                          takes "1" step
```

- Let's change the bounds on both i and j to make both loops faster.

$$T(n) = \sum_{i=1}^{n/2} \sum_{j=1}^{n/2} 1 = \sum_{i=1}^{n/2} (n/2) = n^2/4 \in \Omega(n^2)$$

```
                          * * * * * *
                            * * * * *
          * * *               * * * *
          * * *                 * * *
          * * *                   * *
                                    *
```

Note Pretty Pictures and Faster/Slower are the Same(ish) Picture



- Both the overestimate (upper-bound) and the underestimate (lower-bound) are proportional to n^2

Example 2.5

```
for (i=1; i <= n; i++)
    for (j=1; j <= n; j=j*2)
        sum = sum + 1
```

Time complexity:

a. O(n)
b. O(n lg n)
c. $O(n^2)$
d. $O(n^2 \lg n)$
e. None of these

Example 2.5

```
for (i=1; i <= n; i++)
    for (j=1; j <= n; j=j*2)
        sum = sum + 1
```

j= 1, 2, 4, 8, 16, 32, …

$x <= n < 2x$

$$= 2^0, 2^1, 2^2, \ldots 2^k$$

$2^k <= 2^{\lg n} < 2^{k+1}$

$k <= \lg n < k+1$

$$k = \lfloor \lg n \rfloor$$

$$T(n) = \sum_{i=1}^{n} \sum_{j=0}^{\lfloor \lg n \rfloor} 1 = \sum_{i=0}^{n} \lg n = (n+1)\lg n \in O(n \lg n)$$

Asymptotically flooring doesn't matter

Example 3

```
i = 1
   while i < n do
      for j = 1 to i do
         sum = sum + 1
      i += i
```

Time complexity:

a. $O(n)$
b. $O(n \lg n)$
c. $O(n^2)$
d. $O(n^2 \lg n)$
e. None of these

Example 3

```
i = 1
    while i < n do
       for j = 1 to i do
           sum = sum + 1
       i += i
```

$i = 1, 2, 4, 8, 16, 32, \ldots$

$= 2^0, 2^1, 2^2, \ldots 2^k$

$x <= n\text{-}1 < 2x$

$2^k <= 2^{\lg n-1} < 2^{k+1}$

$k <= \lg n\text{-}1 < k+1$

$k = \lfloor \lg n - 1 \rfloor$

Example 3

```
i = 1
 while i < n do
   for j = 1 to i do
      sum = sum + 1
   i += i
```

Outer loop iteration	i	j	Inner loop iteration
0	1	1-1	1
1	2	1-2	2
2	4	1-4	4
3	8	1-8	8
…	…	…	…
k	2^k	$1\text{-}2^k$	2^k
lg n-1	$2^{\lg n-1}$	$1\text{-}2^{\lg n-1}$	$2^{\lg n-1}$

$$\boxed{\sum_{i=0}^{k} 2^i = 2^{k+1} - 1}$$

$$T(n) = \sum_{i=1}^{\lg n-1} \sum_{j=1}^{2^i} 1 = \sum_{i=1}^{\lg n-1} 2^i = 2^{\lg(n-1)+1} - 1 \in O(n)$$

Example 3 (another approach)

```
i = 1
   while i < n do
      for j = 1 to i do
         sum = sum + 1
      i += i
```

i= 1, 2, 4, 8, 16, 32, …

$$T(n) = 1 + 2 + 4 + \ldots x$$

$$= 2^0 + 2^1 + 2^2 + \ldots 2^k$$

$$T(n) = \sum_{i=1}^{?} 2^i$$

x <= n-1 < 2x

$2^k <= 2^{\lg n\text{-}1} < 2^{k+1}$

k <= lg n-1< k+1

$$k = \lfloor \lg n - 1 \rfloor$$

Example 3 (another approach)

```
i = 1
   while i < n do
      for j = 1 to i do
         sum = sum + 1
      i += i
```

Count this line

$$T(n) = \sum_{i=1}^{\lg(n-1)} 2^i$$

$$T(n) = \sum_{i=1}^{\lg(n-1)} 2^i = 2^{\lg(n-1)+1} - 2 \in O(n)$$

Example 4

- Conditional

```
if C then S_1 else S_2
```

$$O(c) + max\ (\ O(s_1), O(s_2)\ )$$

or

$$O(c) + O(s_1) + O(s_2)$$

- Loops

```
while C do S
```

$$max(O(c), O(s)) * \#\ iterations$$

Example 5

- Recursion almost always yields a *recurrence*
  - Recursive max:

```
int maxRecurse(int nums[], int n){
  if (n== 1)
      return nums[0];

  return max(maxRecurse(nums, n-1), nums[n-1]);
}
```

  - Recurrence relation

$$T(1) \le b \quad \text{if } n \le 1$$
$$T(n) \le c + T(n - 1) \quad \text{if } n > 1$$

Example 5

```
T(1) <= b                 if n <= 1
T(n) <= c + T(n - 1)      if n > 1
```

- **Analysis**
  - `T(n) <= c + c + T(n - 2)` (substitution)
  - `T(n) <= c + c + c + T(n - 3)` (substitution)
  - `T(n) <= kc + T(n - k)` (extrapolating **0 < k ≤ *n***)
  - `T(n) <= (n - 1)c + T(1)` (for **k = *n* - 1**)
  - `T(n)  = (n - 1)c + b`
- `T(n)` $\in$ O(n)

Example 6

- Mergesort algorithm
  - split list in half, sort first half, sort second half, merge together

- Recurrence relation

$$T(1) \le b \qquad \text{if } n \le 1$$
$$T(n) \le 2T(n/2) + cn \qquad \text{if } n > 1$$

Example 6

- Mergesort algorithm

```
T(1) <= b                        if n <= 1
T(n) <= 2T(n/2) + cn             if n > 1
```

Time complexity:

a. $O(n)$
b. $O(n \lg n)$
c. $O(n^2)$
d. $O(n^2 \lg n)$
e. None of these

Example 6

$$
\begin{array}{ll}
T(1) <= b & \textit{if } n <= 1 \\
T(n) <= 2T(n/2) + cn & \textit{if } n > 1
\end{array}
$$

- **Analysis**

  $T(n) <= 2T(n/2) + cn$

  $<= 2(2T(n/4) + c(n/2)) + cn$ (substitution)

  $= 4T(n/4) + cn + cn$

  $<= 4(2T(n/8) + c(n/4)) + cn + cn$ (substitution)

  $= 8T(n/8) + cn + cn + cn$

  $<= 2^k T(n/2^k) + kcn$ (extrapolating $1 < k \le n$)

  $<= nT(1) + cn \lg n$ (for $2^k = n$ or $k = \lg n$)

- $T(n) \in O(n \lg n)$

Example 7

- Recursive Fibonacci

```
if (n == 0 or n == 1)
     return 1
else
     return Fib(n – 1) + Fib(n – 2)
```

- Recurrence relation

$$T(0),\ T(1) >= b$$
$$T(n) >= T(n - 1) + T(n - 2) + c \quad \textit{if } n > 1$$

- Claim

$T(n) >= b\phi^{n-1}$

Where $\phi = (1+\sqrt{5})/2 \approx 1.618$

Note: $\phi^2 = \phi+1$

Example 7

- Claim:

  $T(n) >= b\varphi^{n-1}$

$\phi = (1+\sqrt{5})/2 \approx 1.618$
Note: $\phi^2 = \phi + 1$

- Proof:
  - Basis: $T(0) \geq b > b\phi^{-1}$ and $T(1) \geq b = b\phi^0$
  - Inductive step: Assume $T(m) \geq b\phi^{m-1}$ for all $m < n$

$$\begin{aligned} T(n) &\geq T(n-1) + T(n-2) + c \\ &\geq b\phi^{n-2} + b\phi^{n-3} + c \\ &\geq b\phi^{n-3}(\phi + 1) + c \\ &= b\phi^{n-3}\phi^2 + c \\ &\geq b\phi^{n-1} \end{aligned}$$

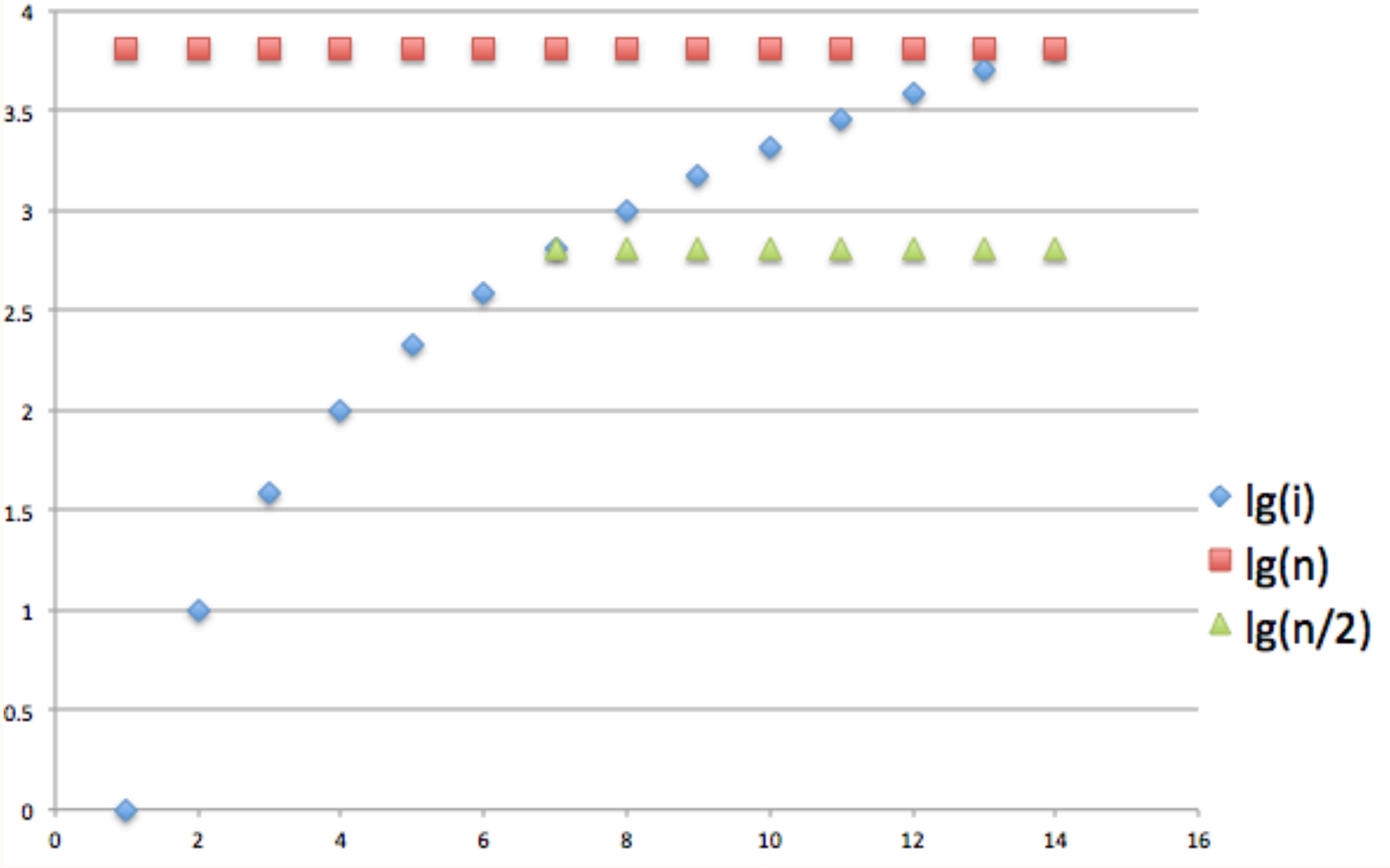
$T(n) \in \Omega(\phi^{n-1})$

Example 8

- Problem: find a tight bound on
  - `T(n) = lg(n!)`

Time complexity:

a. $\Theta(n)$
b. $\Theta(n \lg n)$
c. $\Theta(n^2)$
d. $\Theta(n^2 \lg n)$
e. None of these


4
3.5
3
2.5
2
1.5
1
0.5
0
0
2
4
6
8
10
12
14
16
lg(i)
lg(n)
lg(n/2)

Example 9

- Problem: find a tight bound on
  - `T(n) = lg(n!)`

$$T(n)=\sum_{i=1}^{n}\lg(i)\le\sum_{i=1}^{n}\lg(n)\in O(n\lg n)$$

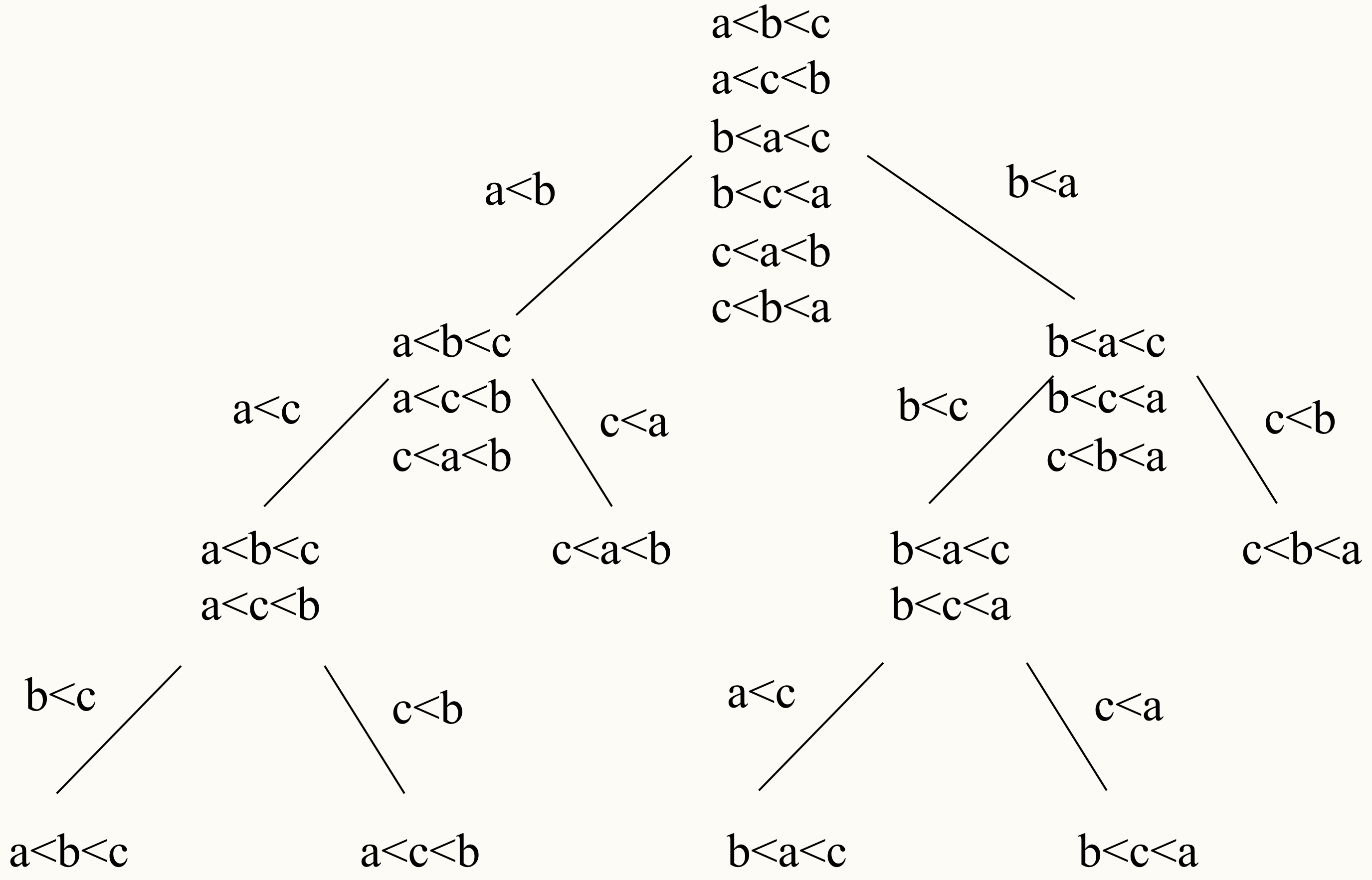
$$T(n)\in\Theta(n\ lg\ n)$$

$$T(n)=\sum_{i=1}^{n}\lg(i)\ge\sum_{i=n/2}^{n}\lg(i)>\sum_{i=n/2}^{n}\lg(n/2)$$

$$\sum_{i=n/2}^{n}\lg(n/2)=n/2(\lg n-1)\in\Omega(n\lg n)$$

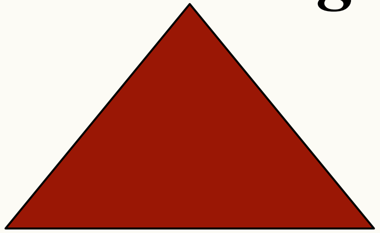
Who Cares About Ω(lg (n!))?

- Let's assume that you want to sort a,b, and c



Who Cares About $\Omega(\lg (n!))$?

- How many ways can you order n items?
  - There are n! possible ways to order them. We could number them 1,2,…n!
- A sorting algorithm must distinguish between these n! choices (because any of them ***might*** be the input).



- So we want a binary tree with n! leaves. A binary tree with height h has 2^h leaves.

$$n! <= 2^h$$
$$lg\ (n!) <= h$$

- So, to distinguish which of the n! orders you were given requires lg(n!) comparisons, which is Ω (n log n)

Learning Goals revisited

- Define which program operations we measure in an algorithm in order to approximate its efficiency.
- Define "input size" and determine the effect (in terms of performance) that input size has on an algorithm.
- Give examples of common practical limits of problem size for each complexity class.
- Give examples of tractable, intractable problems.
- Given code, write a formula which measures the number of steps executed as a function of the size of the input (N).

Continued…

Learning Goals revisited

- Compute the worst-case asymptotic complexity of an algorithm (i.e., the worst possible running time based on the size of the input (N)).
- Categorize an algorithm into one of the common complexity classes.
- Explain the differences between best-, worst-, and average-case analysis.
- Describe why best-case analysis is rarely relevant and how worst-case analysis may never be encountered in practice.
- Given two or more algorithms, rank them in terms of their time and space complexity.

Continued…

Learning Goals revisited

- Define big-O, big-Omega, and big-Theta: $O(\bullet)$, $\Omega(\bullet)$, $\Theta(\bullet)$
- Explain intuition behind their definitions.
- Prove one function is big-O/Omega/Theta of another function.
- Simplify algebraic expressions using the rules of asymptotic analysis.
- List common asymptotic complexity orders, and how they compare.