A Sophomoric Introduction to Shared-Memory Parallelism and Concurrency

Lecture 3
Parallel Prefix, Pack, and Sorting

Steve Wolfman, based on work by Dan Grossman

(with really tiny tweaks by Alan Hu)

Learning Goals

- Judge appropriate contexts for and apply the parallel map, parallel reduce, and parallel prefix computation patterns.

- And also… lots of practice using map, reduce, work, span, general asymptotic analysis, tree structures, sorting algorithms, and more!

Outline

Done:

- Simple ways to use parallelism for counting, summing, finding
- (Even though in practice getting speed-up may not be simple)
- Analysis of running time and implications of Amdahl’s Law

Now: Clever ways to parallelize more than is intuitively possible

- Parallel prefix
- Parallel pack (AKA filter)
- Parallel sorting
  - quicksort (not in place)
  - mergesort

The prefix-sum problem

Given a list of integers as input, produce a list of integers as output where **`output[i]`** = **`input[0]+input[1]+…+input[i]`**

Sequential version is straightforward:

```
Vector<int> prefix_sum(const vector<int>& input){
  vector<int> output(input.size());
  output[0] = input[0];
  for(int i=1; i < input.size(); i++)
    output[i] = output[i-1]+input[i];
  return output;
}
```

Example:

input	**42**	**3**	**4**	**7**	**1**	**10**
output						

The prefix-sum problem

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  for(int i=1; i < input.size(); i++)
    output[i] = output[i-1]+input[i];
  return output;
}
```

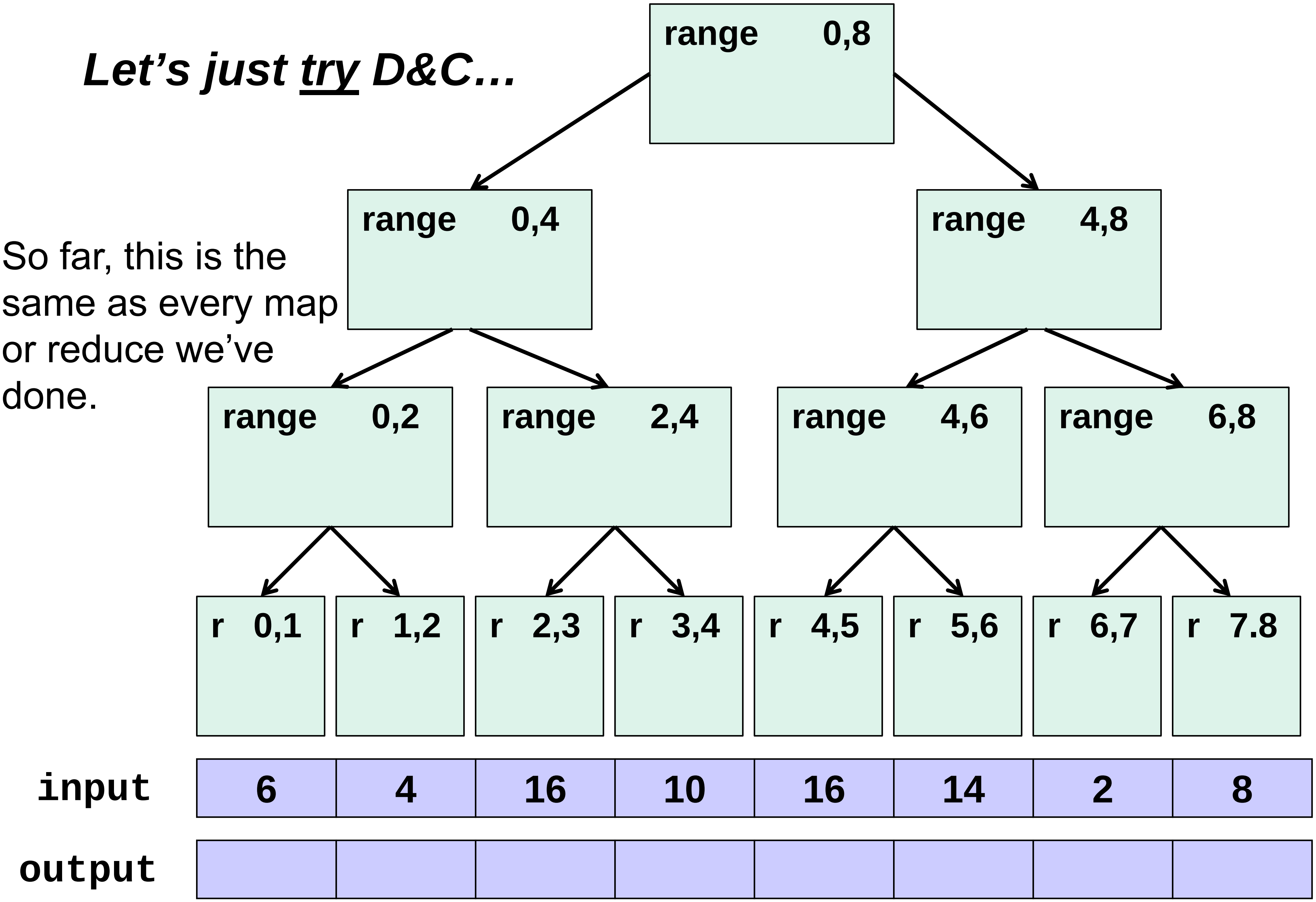
Why isn't this (obviously) parallelizable? Isn't it just map or reduce?

Work:

Span:

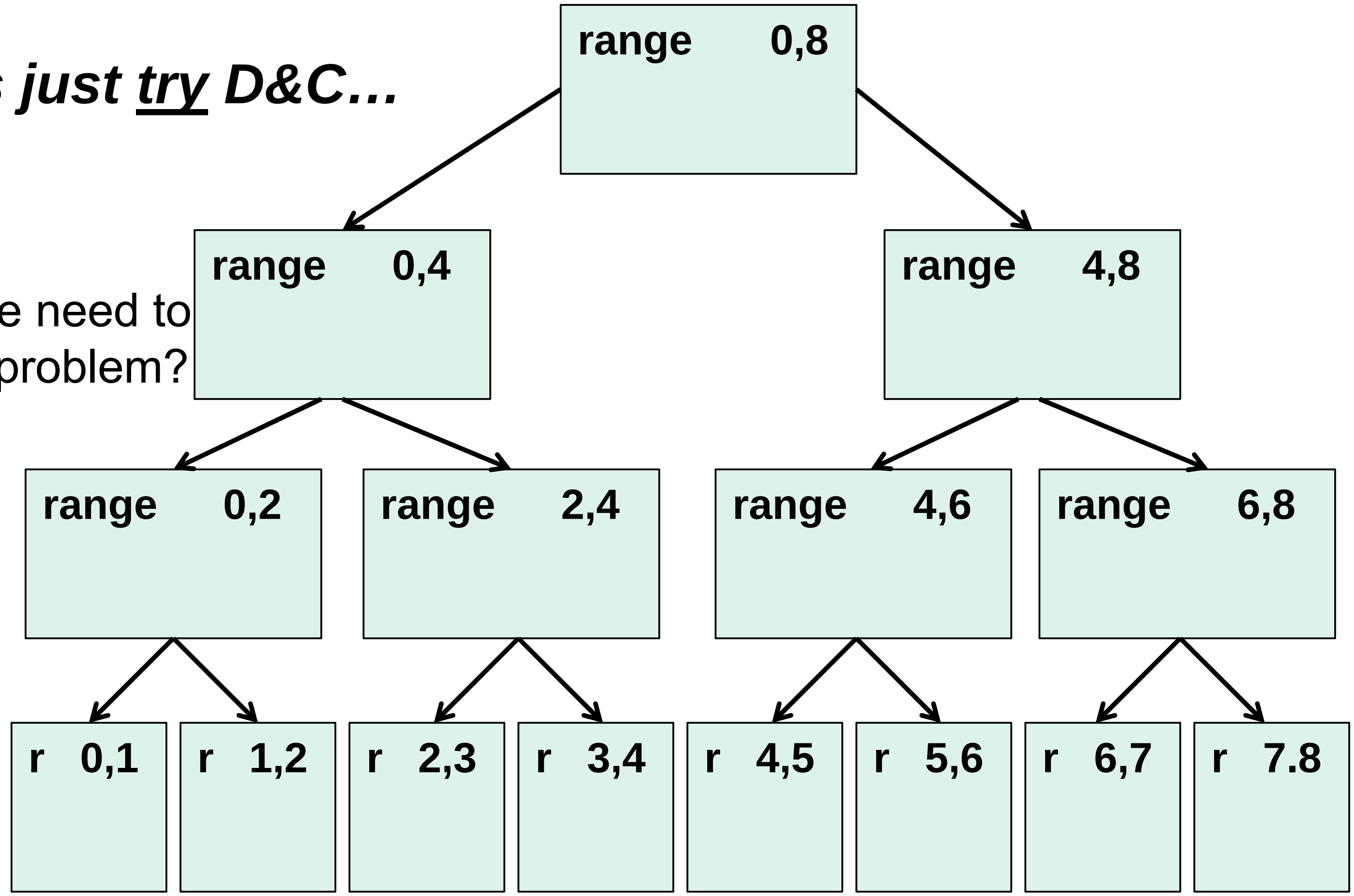
Let's just <u>try</u> D&C…

So far, this is the same as every map or reduce we've done.



Let's just <u>try</u> D&C…

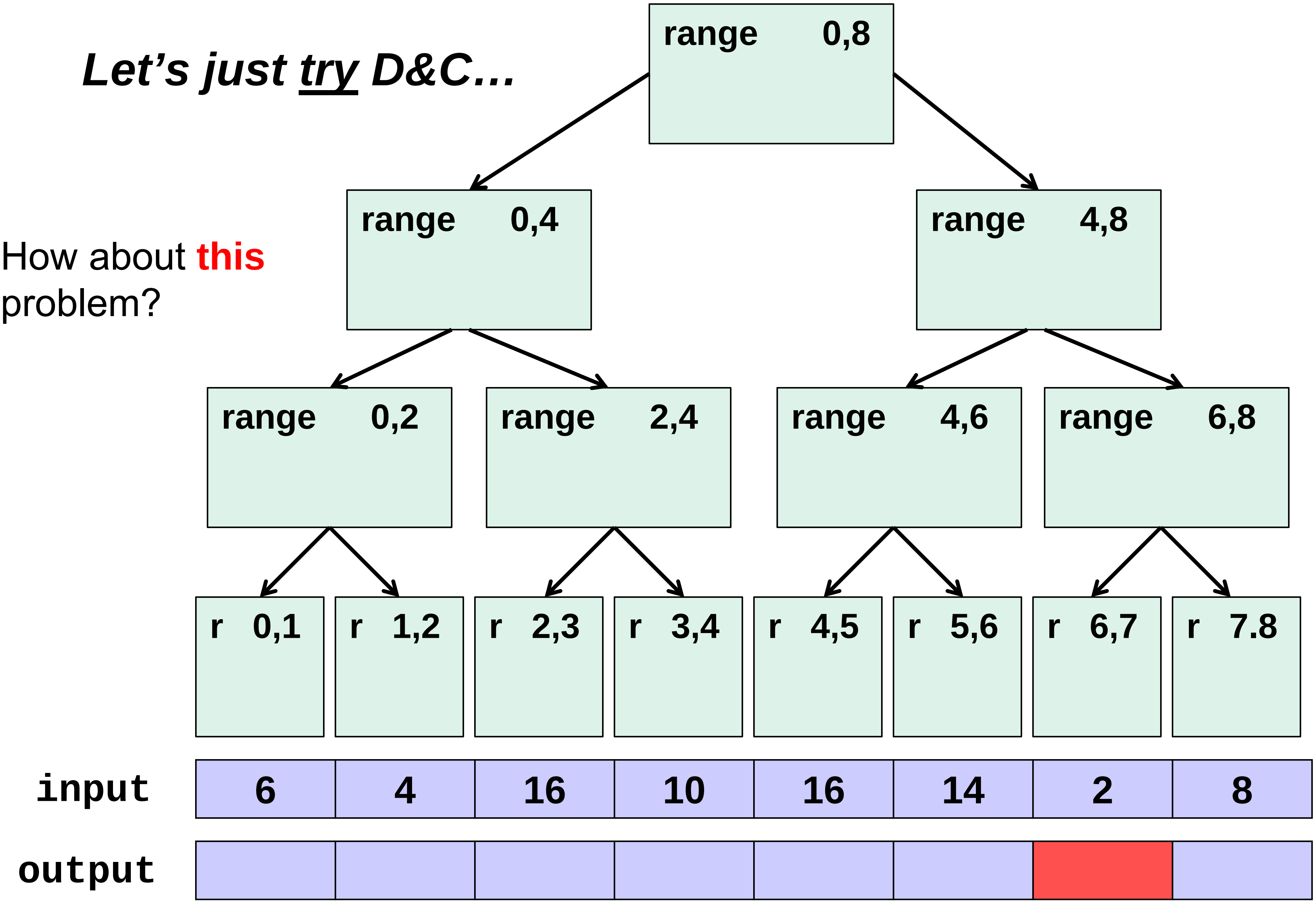
What do we need to solve **this** problem?



`input`	6	4	16	10	16	14	2	8
`output`								

Let's just <u>try</u> D&C…

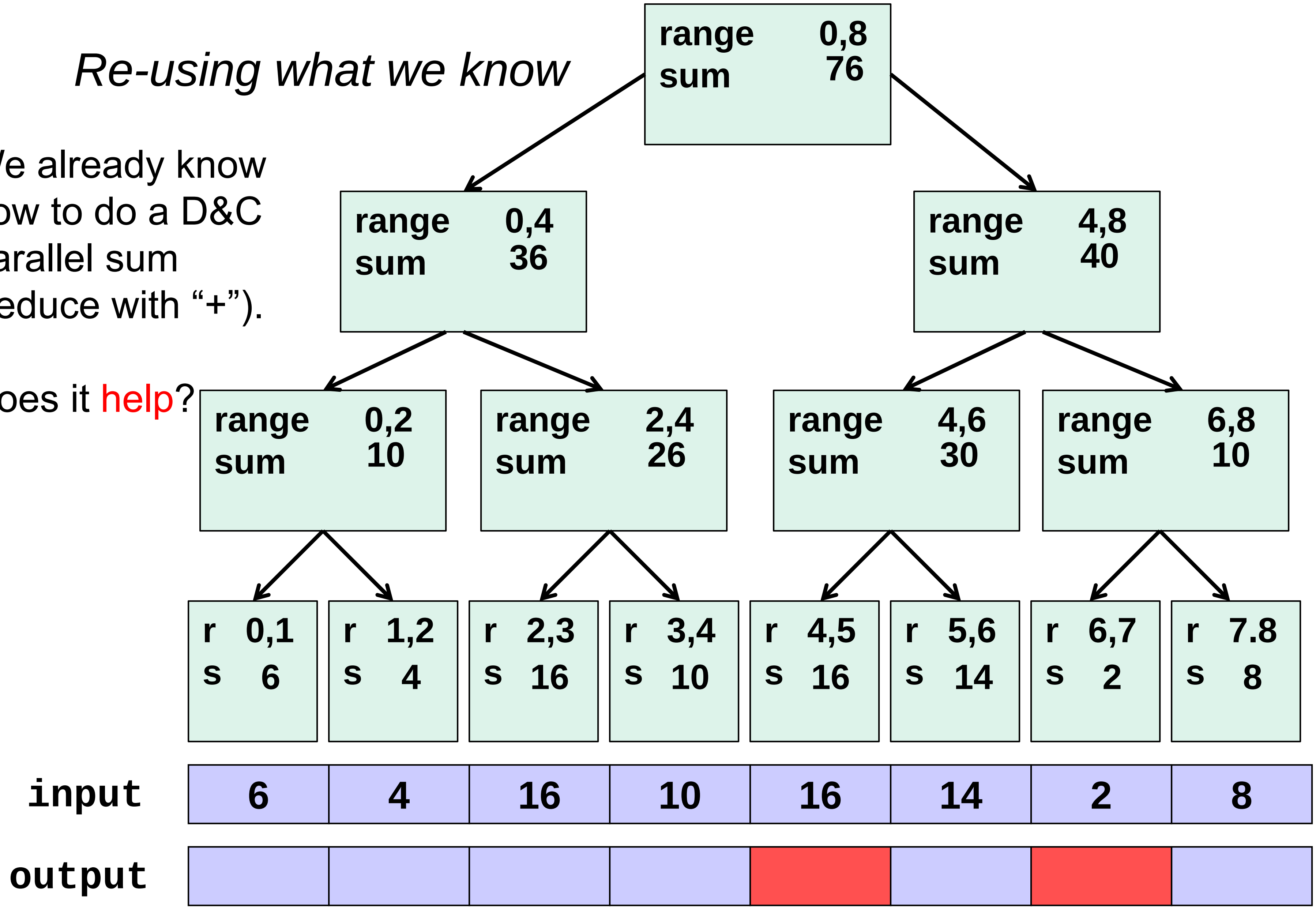
How about **this** problem?



Re-using what we know

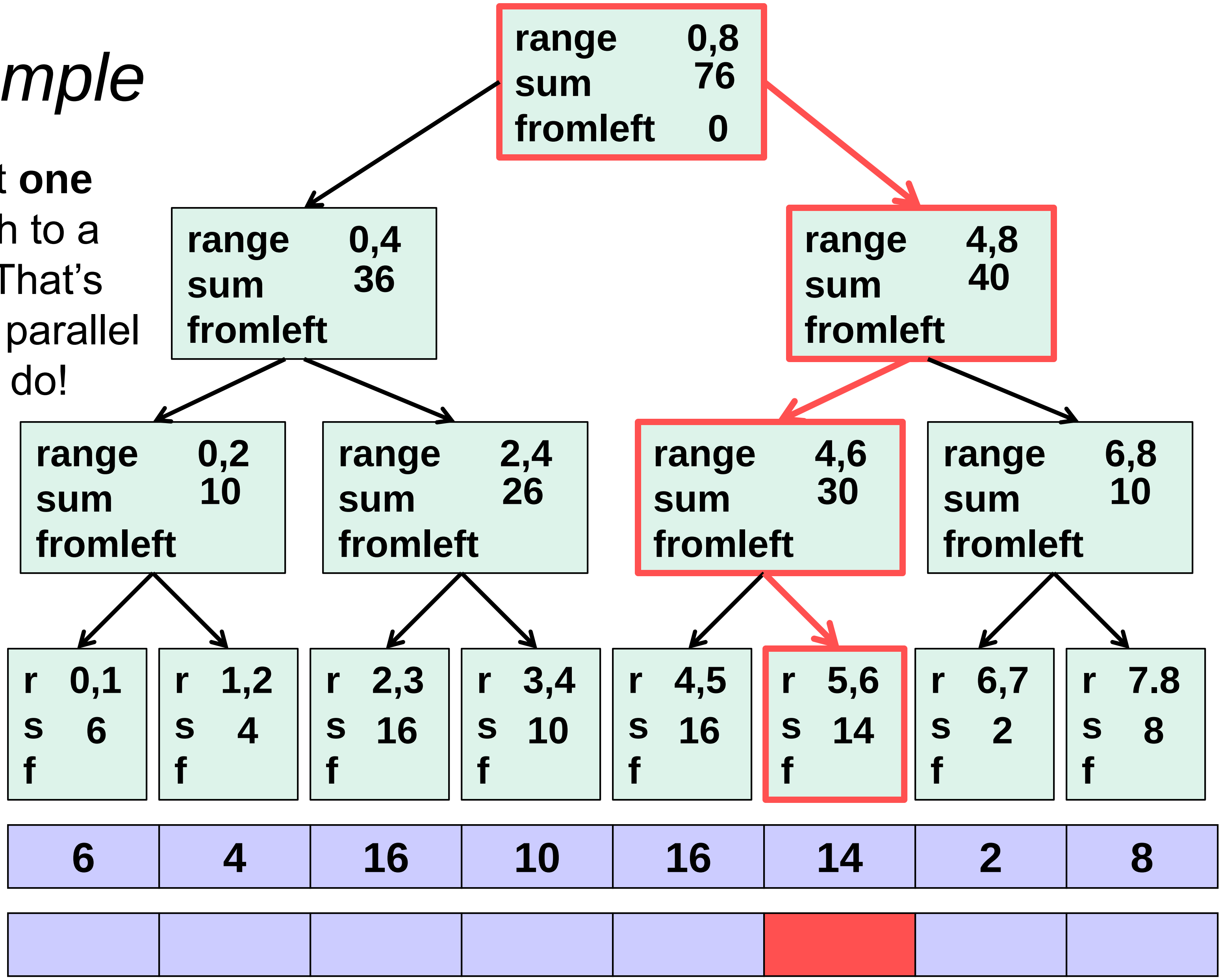
We already know how to do a D&C parallel sum (reduce with "+").

Does it help?



Example

Let's do just **one** branch (path to a leaf) **first**. That's what a fully parallel solution will do!



Algorithm from [Ladner and Fischer, 1977]

Parallel prefix-sum

The parallel-prefix algorithm does two passes:

1. build a “sum” tree bottom-up
2. traverse the tree top-down, accumulating the sum from the left

The algorithm, step 1

1. Step one does a parallel sum to build a binary tree:
   - Root has sum of the range [**`0,n`**)
   - An internal node with the sum of [**`lo,hi`**) has
     - Left child with sum of [**`lo,middle`**)
     - Right child with sum of [**`middle,hi`**)
   - A leaf has sum of [**`i,i+1`**), i.e., **`input[i]`**

How? Parallel sum but explicitly build a tree:

```
return left+right;  ⇒  return new Node(left->sum + right->sum,
                                        left, right);
```

Step 1: Work? Span?

The algorithm, step 2

2. Parallel map, passing down a **`fromLeft`** parameter
   - Root gets a **`fromLeft`** of **`0`**
   - Internal node along:
     - to its left child the same **`fromLeft`**
     - to its right child **`fromLeft`** plus its left child's **`sum`**

   (already calculated in step 1!)
   - At a leaf node for array position **`i`**, **`output[i]=fromLeft+input[i]`**

How? A map down the step 1 tree, leaving results in the output array.

Notice the *invariant*: **`fromLeft`** is the sum of elements left of the node's range

Step 2: Work? Span?

Parallel prefix-sum

The parallel-prefix algorithm does two passes:

1. build a "sum" tree bottom-up
2. traverse the tree top-down, accumulating the sum from the left

Step 1:	Work: ***O***(*n*)	Span: ***O*(lg** *n***)**
Step 2:	Work: ***O***(*n*)	Span: ***O*(lg** *n***)**
Overall:	Work?	Span?

Paralellism (work/span)?

In practice, of course, we'd use a sequential cutoff!

Parallel prefix, generalized

Can we use parallel prefix to calculate the minimum of all elements to the left of **`i`**?

In general, what property do we need for the operation we use in a parallel prefix computation?

Outline

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- Parallel prefix
- Parallel pack (AKA filter)
- Parallel sorting
  - quicksort (not in place)
  - mergesort

Pack

AKA, **`filter`** ☺

Given an array **`input`**, produce an array **`output`** containing only elements such that **`f(elt)`** is **`true`**

Example:
```
input [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
f: is elt > 10
output [17, 11, 13, 19, 24]
```

Parallelizable? Sure, using a list concatenation reduction.

Efficiently parallelizable on arrays?

Can we just put the output straight into the array *at the right spots*?

Pack as map, reduce, prefix combo??

Given an array **`input`**, produce an array **`output`** containing only elements such that **`f(elt)`** is **`true`**

Example: **`input [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]`**

`f: is elt > 10`

Which pieces can we do as maps, reduces, or prefixes?

Parallel prefix to the rescue

1. Parallel map to compute a bit-vector for true elements

```
input  [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
bits   [1,  0, 0, 0,  1, 0,  1,  1, 0,  1]
```

2. Parallel-prefix sum on the bit-vector

```
bitsum [1,  1, 1, 1,  2, 2,  3,  4, 4,  5]
```

3. Parallel map to produce the output

```
output [17, 11, 13, 19, 24]
```

```
output = new array of size bitsum[n-1]
FORALL(i=0; i < input.size(); i++){
  if(bits[i])
    output[bitsum[i]-1] = input[i];
}
```

Pack Analysis

Step 1: Work? Span?
(compute bit-vector with a parallel map)

Step 2: Work? Span?
(compute bit-sum with a parallel prefix sum)

Step 3: Work? Span?
(emplace output with a parallel map)

Algorithm: Work? Span?
Parallelism?

As usual, we can make lots of efficiency tweaks… with no asymptotic impact.

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  - quicksort (not in place)
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Parallelizing Quicksort

Recall quicksort was sequential, in-place, expected time $O(n \lg n)$

	Best / expected case *work*
1. Pick a pivot element	**O(1)**
2. Partition all the data into:	**O(n)**
A. The elements less than the pivot	
B. The pivot	
C. The elements greater than the pivot	
3. Recursively sort A and C	**2T(n/2)**

How do we parallelize this?

What span do we get?

$T_\infty(n) =$

Parallelizing Quicksort

Recall quicksort was sequential, in-place, expected time $O(n \textbf{ lg } n)$

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How good is $O(\lg n)$ *Parallelism?*

Given an infinite number of processors, $O(\lg n)$ faster.

So… sort 10^9 elements 30 times faster?! That's not much ☹

Can't we do better? What's causing the trouble?

(Would using $O(n)$ space help?)

Parallelizing Quicksort

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Parallelizing Quicksort

Recall quicksort was sequential, in-place, expected time $O(n \mathbf{lg}\ n)$

	Best / expected case *span*
1. **Pick a pivot element**	**O(1)**
2. **Partition all the data into:**	**O(log n) parallel pack**
A. **The elements less than the pivot**	
B. **The pivot**	
C. **The elements greater than the pivot**	
3. **Recursively sort A and C**	**T(n/2)**

How do we parallelize this?

What span do we get?

$T_{\infty}(n) =$

Analyzing $T_\infty(n) = \mathbf{lg}\ n + T_\infty(n/2)$

Turns out our techniques from way back at the start of the term will work just fine for this:

$T_\infty(n)$	$= \mathbf{lg}\ n + T_\infty(n/2)$	if $n > 1$
	$= 1$	otherwise

Parallel Quicksort Example

- Step 1: pick pivot as median of three

8	1	4	9	**0**	3	5	2	7	**6**

- Steps 2a and 2c (combinable): pack less than, then pack greater than into a second array
  - Fancy parallel prefix to pull this off not shown

1	4	0	3	5	2				

1	4	0	3	5	2	6	8	9	7

- Step 3: Two recursive sorts in parallel
  (can limit extra space to one array of size n, as in mergesort)

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mergesort

Recall mergesort: sequential, not-in-place, worst-case $O(n \,\mathbf{lg}\, n)$

1.	**Sort left half and right half**	**2T(n/2)**
2.	**Merge results**	**O(n)**

Just like quicksort, doing the two recursive sorts in parallel changes the recurrence for the span to $T(n) = O(n) + 1T(n/2) \in O(n)$

- Again, parallelism is $O(\mathbf{lg}\, n)$
- To do better, need to parallelize the merge
  - The trick won't use parallel prefix this time

Parallelizing the merge

Need to merge two *sorted* subarrays (may not have the same size)

0	1	4	8	9

2	3	5	6	7

Idea: Suppose the larger subarray has *n* elements. In parallel:

- merge the first *n*/2 elements of the larger half with the "appropriate" elements of the smaller half
- merge the second *n*/2 elements of the larger half with the rest of the smaller half

Parallelizing the merge

0	**4**	**6**	**8**	**9**

1	**2**	**3**	**5**	**7**

Parallelizing the merge

0	4	6	8	9

1	2	3	5	7

1. Get median of bigger half: $O(1)$ to compute middle index

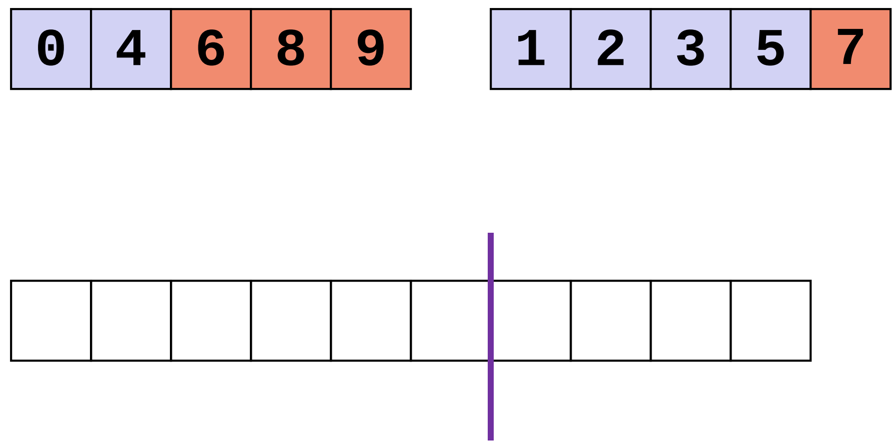
Parallelizing the merge

0	4	6	8	9

1	2	3	5	7

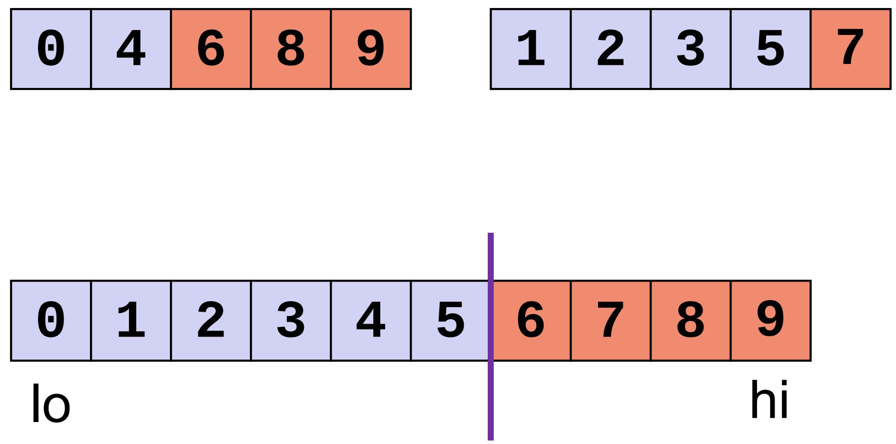
1. Get median of bigger half: $O(1)$ to compute middle index
2. Find how to split the smaller half at the same value as the left-half split: $O($**`lg`** $n)$ to do binary search on the sorted small half

Parallelizing the merge



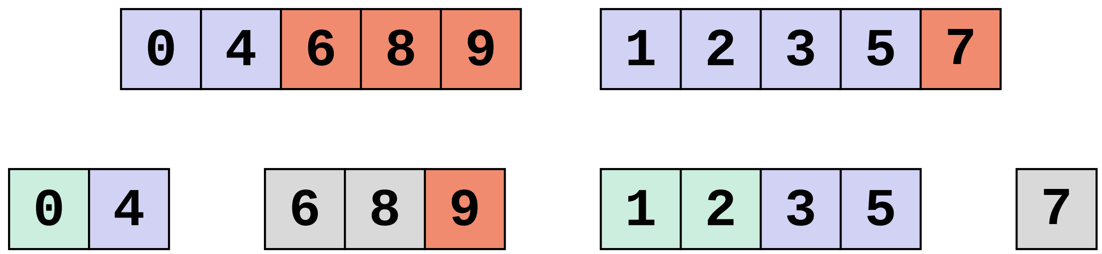
1. Get median of bigger half: $O(1)$ to compute middle index
2. Find how to split the smaller half at the same value as the left-half split: $O($**`lg`** $n)$ to do binary search on the sorted small half
3. Size of two sub-merges conceptually splits output array: $O(1)$

Parallelizing the merge



1. Get median of bigger half: *O*(1) to compute middle index
2. Find how to split the smaller half at the same value as the left-half split: *O*(**`lg`** *n*) to do binary search on the sorted small half
3. Size of two sub-merges conceptually splits output array: *O*(1)
4. Do two submerges in parallel

The Recursion



When we do each merge in parallel, we split the bigger one in half and use binary search to split the smaller one

Analysis

- Sequential recurrence for mergesort:

  $T(n) = 2T(n/2) + O(n)$ which is $O(n \lg n)$

- Doing the two recursive calls in parallel but a sequential merge:

  work: same as sequential   span: $T(n)=1T(n/2)+O(n)$ which is $O(n)$

- Parallel merge makes work and span harder to compute
  - Each merge step does an extra $O(\lg n)$ binary search to find how to split the smaller subarray
  - To merge *n* elements total, do two smaller merges of possibly different sizes
  - But worst-case split is $(1/4)n$ and $(3/4)n$
    - When subarrays same size and "smaller" splits "all" / "none"

Analysis continued

For just a parallel merge of n elements:

- Span is $T(n) = T(3n/4) + O(\mathbf{lg}\ n)$, which is $O(\mathbf{lg}^2\ n)$
- Work is $T(n) = T(3n/4) + T(n/4) + O(\mathbf{lg}\ n)$ which is $O(n)$
- (neither bound is immediately obvious, but "trust me")

So for mergesort with parallel merge overall:

- Span is $T(n) = 1T(n/2) + O(\mathbf{lg}^2\ n)$, which is $O(\mathbf{lg}^3\ n)$
- Work is $T(n) = 2T(n/2) + O(n)$, which is $O(n\ \mathbf{lg}\ n)$

So parallelism (work / span) is $O(n / \mathbf{lg}^2\ n)$

  – Not quite as good as quicksort, but worst-case guarantee
  – And as always this is just the asymptotic result

Looking for Answers?

The prefix-sum problem

Given a list of integers as input, produce a list of integers as output where **`output[i]`** = **`input[0]+input[1]+…+input[i]`**

Sequential version is straightforward:

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Vector<int> prefix_sum(const vector<int>& input){
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  output[0] = input[0];
  for(int i=1; i < input.size(); i++)
    output[i] = output[i-1]+input[i];
  return output;
}
```

Example:

input	42	3	4	7	1	10
output	42	45	49	56	57	67

The prefix-sum problem

Given a list of integers as input, produce a list of integers as output where **`output[i]`** = **`input[0]+input[1]+…+input[i]`**

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  return output;
}
```

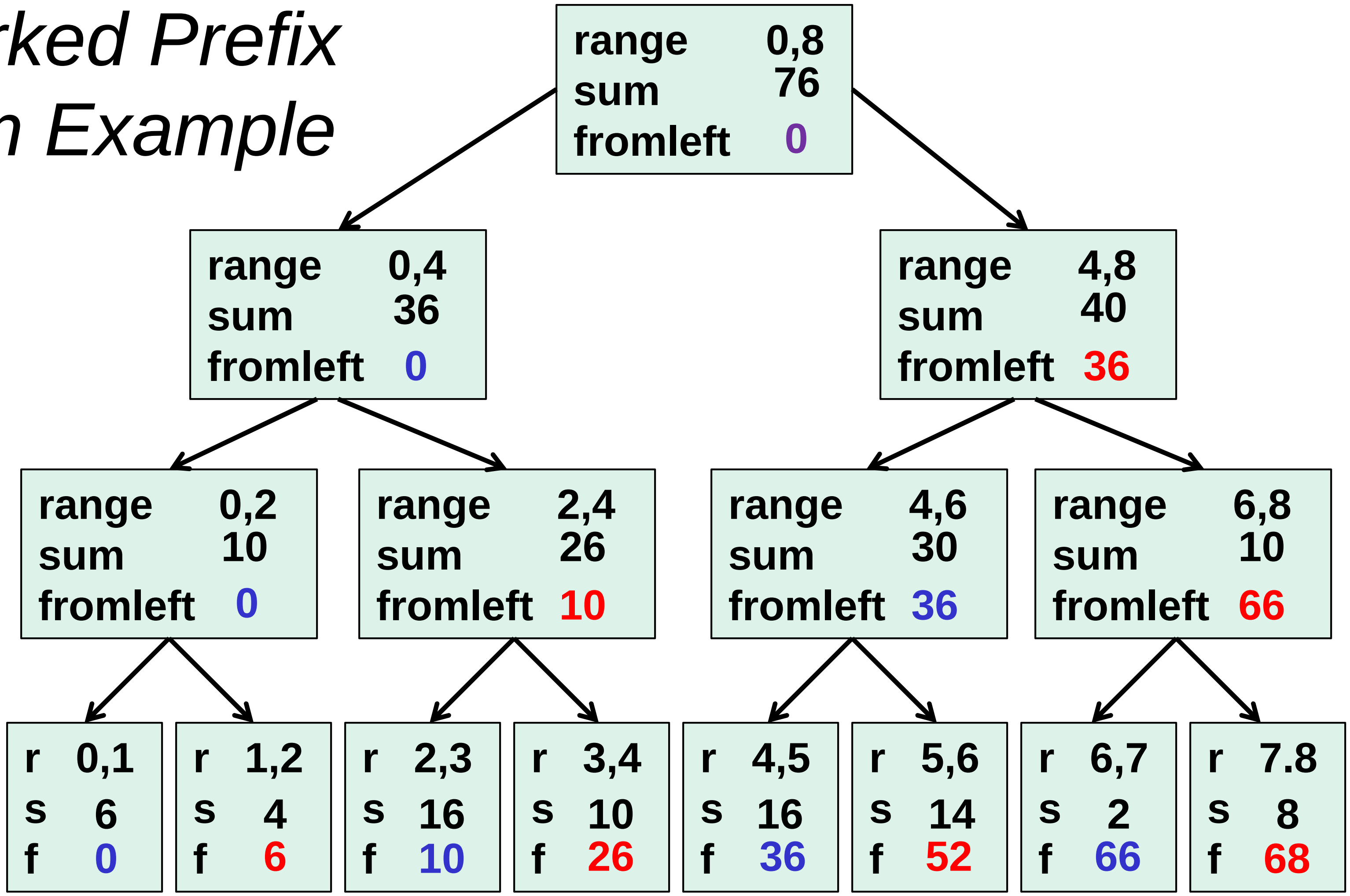
Why isn't this (obviously) parallelizable? Isn't it just map or reduce?

Work: O(n)

Span: O(n) b/c each step depends on the previous.

Joins everywhere!

Worked Prefix Sum Example



`input`	6	4	16	10	16	14	2	8
`output`	6	10	26	36	52	66	68	76

Parallel prefix-sum

The parallel-prefix algorithm does two passes:

1. build a "sum" tree bottom-up
2. traverse the tree top-down, accumulating the sum from the left

Step 1:	Work: $O(n)$	Span: $O(\lg n)$
Step 2:	Work: $O(n)$	Span: $O(\lg n)$
Overall:	Work: $O(n)$	Span? $O(\lg n)$

Paralellism (work/span)? $O(n/\lg n)$

In practice, of course, we'd use a sequential cutoff!

Parallel prefix, generalized

Can we use parallel prefix to calculate the minimum of all elements to the left of **`i`**?

Certainly! Just replace "sum" with "min" in step 1 of prefix and replace fromLeft with a fromLeft that tracks the smallest element left of this node's range.

In general, what property do we need for the operation we use in a parallel prefix computation?

ASSOCIATIVITY! (And not commutativity, as it happens.)

Pack Analysis

Step 1: Work: $O(n)$ Span: $O(\lg n)$

Step 2: Work: $O(n)$ Span: $O(\lg n)$

Step 3: Work: $O(n)$ Span: $O(\lg n)$

Algorithm: Work: $O(n)$ Span: $O(\lg n)$

Parallelism: $O(n/\lg n)$

As usual, we can make lots of efficiency tweaks… with no asymptotic impact.

Parallelizing Quicksort

Recall quicksort was sequential, in-place, expected time $O(n \lg n)$

	Best / expected case *work*
1. Pick a pivot element	**O(1)**
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A. The elements less than the pivot	
B. The pivot	
C. The elements greater than the pivot	
3. Recursively sort A and C	**2T(n/2)**

How should we parallelize this?

Parallelize the recursive calls as we usually do in fork/join D&C.

Parallelize the partition by doing two packs (filters) instead.

Parallelizing Quicksort

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B. The pivot	
C. The elements greater than the pivot	
3. Recursively sort A and C	**2T(n/2)**

How do we parallelize this? First pass: parallel recursive calls in step 3.

What span do we get?

$$T_\infty(n) = kn + T(n/2) = kn + kn/2 + T(n/4) =$$
$$kn/1 + kn/2 + kn/4 + kn/8 + \ldots + 1 \in \Theta(n)$$

Analyzing $T_\infty(n) = \mathbf{lg}\ n + T_\infty(n/2)$

Turns out our techniques from way back at the start of the term will work just fine for this:

$T_\infty(n)$	$= \mathbf{lg}\ n + T_\infty(n/2)$	if $n > 1$
	$= 1$	otherwise

We get a sum like:

$$\mathbf{lg}\ n + \mathbf{lg}\ n - 1 + \mathbf{lg}\ n - 2 + \mathbf{lg}\ n - 3 + \ldots + 1$$

Let's replace **lg** n by **k**:

$$\mathbf{k} + \mathbf{k} - 1 + \mathbf{k} - 2 + \mathbf{k} - 3 + \ldots + 1$$

That's our "triangle" pattern: $O(k^2) = O((\lg n)^2)$